PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. IX. 1876–77. No. 98.

Ninety-Fourth Session.

Monday, 19th February 1877.

DAVID STEVENSON, Esq., Vice-President, in the Chair.

The Chairman reported that the Council had awarded the Makdougall-Brisbane Prize for the biennial period 1874–76, to Alexander Buchan, M.A., for his paper "On the Diurnal Oscillation of the Barometer," which forms one of an important series of Contributions by Mr Buchan to the Advancement of Meteorological Science.

The following Communications were read:—

1. On the Action of Sulphuretted Hydrogen on the Hydrate and on the Carbonate of Trimethyl-sulphine. By Professor Crum Brown.

(*Abstract.*)

The investigation, of which this paper contains the first part, was undertaken with the view of throwing light on the constitution of salts of trimethl-sulphine.

These salts have been represented in two different ways—1st, as compound of tetratomic sulphur, and 2d, as molecular combinations of sulphide of methyl with methylic ethers, just as the ammonia salts have been represented as compounds of pentad nitrogen, or as molecular combinations of ammonia with hydric salts.

It appeared to the author that facts having an important bearing on the question, which of these is the better representation of the constitution of such bodies, might be obtained by the study of the sulphur compounds of trimethyl-sulphine, of the corresponding selenium compounds, and of the intermediate substances.

The following list, in which each substance is formulated—*a*, on the assumption of tetrad sulphur and selenium, and *b*, on the supposition of molecular union illustrates this idea.

	a.		*b.*
(1.)	$(CH_3)_3S{-}SH$	...	$(CH_3)_2S$, CH_3HS.
(2.)	$((CH_3)_3S)_2S$	...	$(CH_3)_2S$, $(CH_3)_2S$, $(CH_3)_2S$.
(3.)	$(CH_3)_3Se{-}SH$	...	$(CH_3)_2Se$, CH_3HS.
(4.)	$((CH_3)_3Se)_2S$	...	$(CH_3)_2Se$, $(CH_3)_2Se$, $(CH_3)_2S$.
(5.)	$\left.\begin{matrix}(CH_3)_3S\\(CH_3)_3Se\end{matrix}\right\}S$	...	$(CH_3)_2S$, $(CH_3)_2Se$, $(CH_3)_2S$.
(6.)	$(CH_3)_3S{-}SeH$	...	$(CH_3)_2S$, CH_3HSe.
(7.)	$((CH_3)_3S)_2Se$	...	$(CH_3)_2S$, $(CH_3)_2S$, $(CH_3)_2Se$.
(8.)	$(CH_3)_3Se{-}SeH$	...	$(CH_3)_2Se$, CH_3HSe.
(9.)	$((CH_3)_3Se)_2Se$	...	$(CH_3)_2Se$, $(CH_3)_2Se$, $(CH_3)_2Se$.
(10)	$\left.\begin{matrix}(CH_3)_3S\\(CH_3)_3Se\end{matrix}\right\}Se$	...	$(CH_3)_2S$, $(CH_3)_2Se$, $(CH_3)_2Se$.

Of these it is obvious that (3) and (6), (5) and (7), and (4) and (10) form three pairs of isomers.

Upon any theory (3) and (6) may be expected to be different, but it is not so with the other two pairs. They ought to be different, if the assumption of tetrad sulphur and selenium is correct, but on the theory of molecular combination it is difficult to see how a difference of properties could be accounted for.

The hydrate and also the carbonate of trimethyl-sulphine are readily acted upon in the dry condition by sulphuretted hydrogen, and the product is colourless if air has been rigidly excluded. The characteristic reaction with nitroprusside shows the product to be a sulphide or sulphhydrate, but the action of oxygen upon it is so rapid that it has not yet been obtained in a condition fit for analysis. Before attempting to prepare it in a state of purity, it was thought best to examine the products of its oxidation. These are—(1), an orange polysulphide, which in the presence of moisture

and oxygen is further oxidised, with separation of yellow sulphur, yielding (2) hyposulphite of trimethyl-sulphine.

The action is thus similar to that of sulphuretted hydrogen on potash or carbonate of potash, but takes place with much greater rapidity.

The examination of the sulphide and polysulphide of trimethyl-sulphine will form the subject of the next part of the paper.

2. On Links. By Professor Tait.

(*Abstract.*)

Though in my former papers on knots I have made but little allusion to cases in which two or more closed curves are linked together, the method I have employed is easily and directly applicable to them. I stated to the British Association that the number of intersections passed through in going continuously along a curve, from any intersection to the same again, is always even—whether it be linked with other curves or not. Hence, even when a number of closed curves are linked together, the intersections may be so arranged as to be alternately over and under along each of the curves.

When this is done, each of the meshes has all its angles right or left handed; so that Listing's type-symbols may be employed, just as for a single knotted curve. The scheme, however, consists of as many parts as there are intersecting curves—each part containing, along with each of its own crossings *twice*, each of its intersections with other curves *once*.

Thus AB | A AB | A

or

$$\left.\begin{matrix} 2r^2 \\ 2l^2 \end{matrix}\right\}$$

represents a couple of ovals linked together.

When three ovals are joined, so as to form an endless chain, we have

ABCD | A DCEF | D FEBA | F

or

$$\left.\begin{matrix} 2r^3 + 3r^2 \\ 3l_4 \end{matrix}\right\}$$

Of course such figures can be transformed or deformed according to the methods given in my first paper—the scheme and the type-symbol alike remaining unaltered. And alterations of both scheme and symbol are, in various classes of cases, producible by the processes of my last paper without any change of links or linking.

The genesis of the scheme of a link may be most easily studied by forming a knot into a link. This is done by cutting both turns of the wire at any junction, and joining them again so as to make two closed curves instead of one. No intersections are lost by this process, except that which was cut across, provided, of course, that the original knot had no nugatory intersections, and that none are rendered nugatory by the operation of cutting the whole across.

Any crossing with four adjacent crossings when the turns of the coil pass alternately over and under one another will appear in a scheme as follows :—

$$\begin{array}{ccccccccc} \ldots & A & X & B & \ldots\ldots & C & X & D & \ldots \\ & - & + & - & & + & - & + & \end{array}$$

implying that from X through B and C back to X forms one continuous circuit; similarly from X through D and A back to X.

There are but two ways in which continuity can be kept up if we cut the cord twice at X, and reunite the ends in a different arrangement from the original one.

It is obvious that if we pass from C to B, by way of X (abolished), and similarly with the rest, we divide the continuous closed curve into two separate (but generally inter-linked) closed curves. If we pass from A, by way of X (abolished) to C, we pass thence in time to B, and finally by way of D to A. Thus the curve remains continuous, but with one intersection less than at first. And, in either case, the alternate order of the signs of the crossings will be maintained throughout.

In the former of these modes we take the part containing C and B (and we may, if we please, also take the rest) in the same order as before the change. The scheme is therefore, without any other change, simply divided into two parts, which are separated from one another by the (abolished) junction X in its two positions.

In the second mode, it is obvious that the letters in one of the two parts separated from one another by the mark X in its two places are simply to be inverted in order without change.

The process presents no difficulties, so that I shall give only two simple examples. Thus the scheme of the pentacle, viz. :—

$$\mathrm{A\,D\,B\,E\,C\,A\,D\,B\,E\,C} \mid \mathrm{A}$$

is divided at A (in this case it does not matter which junction we take) into the two superposed non-autotomic ovals

$$\mathrm{D\,B\,E\,C} \mid \mathrm{D}, \qquad \mathrm{D\,B\,E\,C} \mid \mathrm{D},$$

by the first mode :—, and is simplified into

$$\mathrm{D\,B\,E\,C\,C\,E\,B\,D} \mid \mathrm{D}$$

(*i.e.*, a wholly nugatory scheme) by the second.

The type-symbols in the original state, and in the two altered states, are, respectively,

$$\left.\begin{matrix} 2r^5 \\ 5l^2 \end{matrix}\right\}$$

$$\left.\begin{matrix} 2r^4 \\ 4l^2 \end{matrix}\right\}$$

$$\left.\begin{matrix} r^8 \\ 3l^2+2l \end{matrix}\right\}$$

The last of these is virtually nothing. In fact, terms in r or l to the first power are rejected by Listing. And, when these loops are taken off by untwisting or by opening up, the scheme becomes

$$\left.\begin{matrix} r^4 \\ l^2+2l \end{matrix}\right\}$$

and a second application of the process removes the whole.

Operating in a similar way upon the only other figure with five non-nugatory intersections—viz. :—

$$\mathrm{A_4D_4B_2E_2C_2A_4D_4C_2E_2B_2} \mid \mathrm{A}$$

or

$$\left.\begin{matrix} 2r^4+r^2 \\ 2l^3+l^2 \end{matrix}\right\}$$

we find three classes of cases, according to the particular intersection operated on.

[I may here introduce, though it involves a slight digression, a method which I have found very convenient as an assistance in finding which intersections have similar properties as regards the

figures which will be obtained when they are made in turn the point of section. In the scheme above written the suffixes express the numbers of letters which intervene, in the scheme, between the two appearances of the same letter. If n be the whole number of letters, the suffix may of course be either $2r$ or $2n-2r-2$. It is convenient to write always that one of these two numbers which is not greater than the other. When a particular suffix occurs only once, the corresponding crossing has evidently different properties from the others; if twice, we find in general that the corresponding crossings have similar properties. If three times, two of them have usually like properties, but the third not—and so on. This method is useful, but it is in certain cases misleading. In fact, we must look not only at the suffix itself, but at the place which it occupies relatively to the whole group of suffixes, in order to obtain absolutely definite information. Something similar to this is obviously hinted at in Listing's paper, where he seems to determine the number of possible transformations of the figure representing a symbol, by treating the numerical coefficients much as I have here treated the suffixes. But this is mere conjecture on my part.]

By this method then, or by examining the diagram, we see that A and D are similar, so are B and C, while E may possibly possess distinct properties of its own. We need, therefore, take only three cases, A, B, and E.

a.) Divide at A. Then we have either

$$\mathrm{DBEC} \mid \mathrm{D} \quad \mathrm{DCEB} \mid \mathrm{D}$$

$$\left.\begin{matrix} 2r^4 \\ 4l^2 \end{matrix}\right\}$$

two ovals crossing one another, one taken right handed, the other left; or

$$\mathrm{DBECBECD} \mid \mathrm{D} = \mathrm{BECBEC} \mid \mathrm{B}$$

$$\left.\begin{matrix} 2r^3 \\ 3l^2 \end{matrix}\right\}$$

the trefoil knot; for D becomes nugatory.

b.) Divide at B. We have either

$$\mathrm{AD} \mid \mathrm{A} \qquad \mathrm{ECADCE} \mid \mathrm{E} = \mathrm{DA} \mid \mathrm{D}$$

$$\left.\begin{matrix} 2r^2 \\ 2l^2 \end{matrix}\right\}$$

two linked ovals, C and E having become nugatory; or

ECADCEDA | E

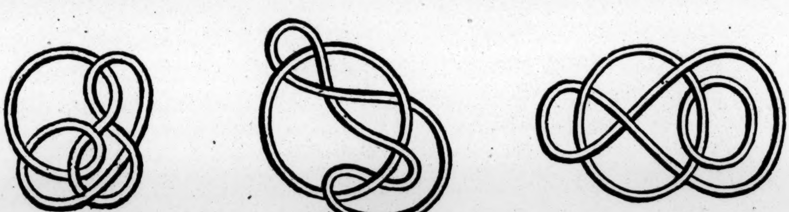
$$\left.\begin{matrix}2r^3+r^2\\2l^3+\ l^2\end{matrix}\right\}$$

an amphicheiral knot, the only knot with 4 intersections.

c.) Dividing at E we find the same results as for B and C.

From the rules just given for removing an intersection, it is of course easy to pass to those required for the introduction of a new intersection.

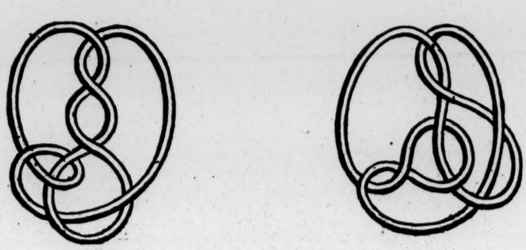
In endeavouring to frame a general method of determining whether a particular type-symbol necessarily denotes one continuous curve, or a superposition of two or more curves, I was completely unsuccessful. But, as indicated in a note to my last paper, I found the reason to be that *no such distinction necessarily exists.* And by the application of the methods of adding or removing intersections given, I found a number of instances in which the same type-symbol may represent many entirely different kinds of figures. Thus the following



are all represented alike by the symbol

$$\left.\begin{matrix}r^5+r^4+r^3+r^2\\l^4+2l^3+2l^2\end{matrix}\right\}$$

But I have since succeeded in obtaining cases in which the same type-symbol represents two perfectly distinct single closed curves. One instructive example is the following



The common type-symbol is

$$\left.\begin{array}{l} r^5 + r^4 + r^3 + 2r^2 \\ l^5 + l^4 + l^3 + 2l^2 \end{array}\right\}$$

But the schemes are

$$A_6E_6B_4G_6C_6H_6D_6B_4E_6A_6F_2C_6G_6F_2H_6D_6 \mid A$$

and

$$A_6D_4B_2H_4C_6F_4D_4A_6E_4G_2F_4C_6G_2E_4A_4B_2 \mid A$$

Now no change in lettering can affect the suffixes, so that the two schemes are essentially different. In fact the sum of the suffixes is 84 in the first scheme, but only 64 in the second. The first has only one degree of beknottedness, the second has two. The first is not amphicheiral, the second is.

There is no connection between the type-symbol, as Listing gives it, and the singleness or complexity of the curve represented, but it is possible to make analogous symbols capable of expressing everything of this kind. Only we must now adopt something very much resembling Crum Brown's Graphical Formulæ for chemical composition. Some very remarkable relations follow from this process, but I can only allude to a few of the simpler of them in this abstract.

The only necessary relations among the numbers forming the right or left part of a symbol are satisfied if no one is greater than the sum of the others, and if the sum of all is even. With any set of numbers subject to these conditions, we can form the right or left-hand side of a symbol—and from that we can form the other when we know the grouping.

An example or two will make this clear. Take, for instance, the symbol

$$\left.\begin{array}{l} 2r^4 + r^2 \\ 2l^3 + l^2 \end{array}\right\}$$

which represents the five-crossing knot of p. 242 above.

A glance at the figure shows that the following is the arrangement of the right-handed meshes.

$$\begin{array}{ccc} & r^2 & \\ / & & \backslash \\ r^4 & \equiv & r^4 \end{array}$$

the single mesh with two corners having one of these corners in common with each of the two four-sided meshes, which again

have three corners in common. Hence in this notation *the joining lines represent the crossings.* Hence also the characters of the left-hand meshes are obvious from the figure. Outer space has the three external lines for corners—inside there is one triangle and two spaces bounded by two lines each (*i.e.*, with two corners). Thus we reproduce the left-hand part of Listing's symbol. But the figure also shows us which lines (corners) each pair of these has in common, and enables us at once to draw the annexed figure

$$\left(\begin{array}{c} l^3 \\ | \\ l^2 \\ | \\ l^2 \\ | \\ l^3 \end{array}\right)$$

which gives us exactly the same information as the first, only from a different point of view.

The connections in the former figure cannot be varied, so that, in this particular case, Listing's symbol for the right-handed meshes alone suffices to draw the figure; at least if nugatory crossings be rejected. Such would arise, for instance, if we tried to draw the symbol in the form

$$\begin{array}{c} r^2 \\ \| \\ r^4 \\ \| \\ r^4 \\ \mathrm{O} \end{array}$$

which would give three ovals joined like the links of a chain—the last having an internal nugatory loop. In this case the second part of the symbol would be

$$l^5 + 2l^2 + l$$

where the nugatory character of one intersection is clearly exhibited.

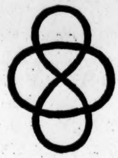
But, if we had merely the left-hand part of the symbol given us, we might adjust it thus

$$\begin{tikzcd} & l^2 \arrow[dl, dash] \arrow[dr, dash] & \\ l^3 \arrow[rr, dash] \arrow[dr, dash] & & l^3 \arrow[dl, dash] \\ & l^2 & \end{tikzcd}$$

which would correspond to

$$r^4 + 2r^3$$

for the right-handed part, and would give us the form

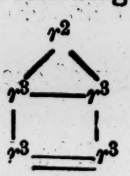


or one of its deformations.

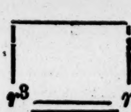
The criterion by which to distinguish at once whether such symbolic representations as those just given represent knots or links is easy to find. If we remember that each of the (even number of) crossings lying on a closed curve is a corner of one black and of one white mesh (contained within the curve)—while each of the crossings lying within it is a corner of each of two white and of two black meshes —we see that unless we can enclose a part of the graphic symbol in such a way that the sum of the exponents within the enclosure, and that formed by the doubling of the number of the joining lines which are wholly within the enclosure, and adding it to the number of those which cut the boundary, are *equal even numbers*—the figure is necessarily a knot. But if we can enclose such a part, it requires to be farther examined to test whether the figure consists of links or is a single knot.

Thus, in the example just given, the part

is a simple oval divided by two intersecting chords into three-cornered meshes—but in the following formula



although the par

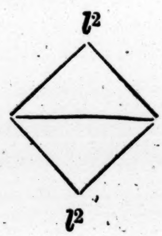


seems to fulfil the conditions above, it does not represent a separ-

ate closed curve. In fact, the upper line represents a crossing *on the boundary*, at which there is (internally) only a left-handed mesh, which is impossible if the boundary were a closed curve.

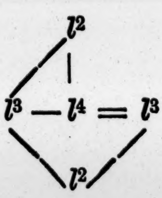
And the lowest line in the figure is a point in the boundary which forms a common vertex of *three* (internal) meshes, two right and one left-handed. This, also, is inconsistent with the boundary's being a closed curve.

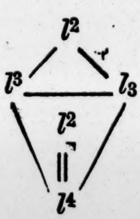
There is only one other case which may cause a little trouble. It can easily be seen by the fig. of last page. For we may take out the following part of the symbol



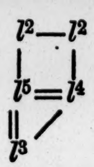
which must obviously represent the lemniscate in the figure. Its exponents and lines do not satisfy our condition: but they will do so if we remove the diagonal line—which corresponds to what is (in the lemniscate when alone) a nugatory intersection.

I conclude by giving the representations, according to the method just explained, of some of the preceding figures. Thus the three first figs. of p. 325 are, respectively,

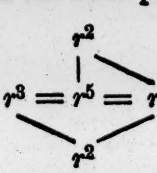




while the pair of common-symbol knots on the same page are



and



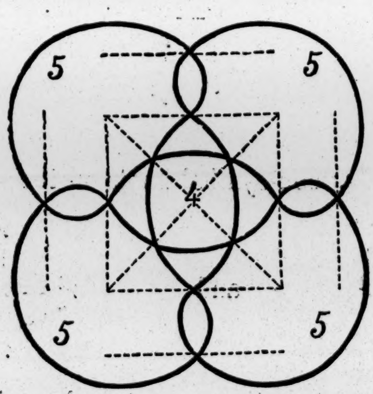
It may be observed that the present method gives great facilities for the study of cases in which knots are reduced, or are changed

into links, by the removal of an intersection. For, to take off an intersection is easily seen to be equivalent simply to rubbing out one connecting line in the figure, and simultaneously diminishing by unity each of the exponents at its ends. If it be the only line connecting these exponents, they are (after reduction by unit each), to be added together. And this consideration enables us to obtain, even more simply than before, the rules for distinguishing a knot from a link. I propose, when I have sufficient leisure, to re-investigate the whole subject from this point of view.

Meanwhile I may notice that it is exceedingly easy to draw the outline of any knot or link by this method. All that is necessary is to select a point in each of the lines in the figure, and join (two and two) all these points which are in the boundary of each closed area. The four lines which will thus be drawn to each of the chosen points must be treated as pairs of continuous lines *intersecting* at these points, and at these only. When there are only two sides—and, therefore, only two points—in an area, two separate lines must be drawn between them, and these must *cross* one another at each of the two points.

The annexed diagram shows the result of this process as applied to the following symbol

$$\begin{array}{ccccc} r^5 & & = & & r^5 \\ & \diagdown & & \diagup & \\ \| & & r^4 & & \| \\ & \diagup & & \diagdown & \\ r^5 & & = & & r^5 \end{array}$$



This method also clears up in a remarkable manner the whole

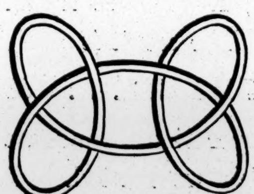
subject of change of scheme of a given knotting which was discussed in my last paper. To give only a very simple instance, notice that the first of the changes there mentioned is merely that from

$$\begin{array}{ccc} & 2 & \\ / & & \backslash \\ m & - & n \\ \| & & \| \end{array} \qquad \text{to} \qquad \begin{array}{ccccc} & m & - & n & \\ \| & \backslash & 2 & / & \| \end{array}$$

where the double lines may stand for any numbers of connection whatever.

I conclude by stating, in illustration of the remarks made at the end of my last paper, that I have hastily (though I hope correctly) investigated the nature of all the valid combinations among 720 which are possible in the even places of a scheme corresponding to 6 intersections (only 80 of these are not obviously nugatory) —and that I find *only four really distinct forms.* They are

1. Two separate trefoil knots. Here there are two degrees of beknottedness.



2. The amphicheiral form. (Figured on p. 295 of my *Note on Beknottedness.* Also in a clear form in the last cut of my first paper.)

3. Fig. p. 297 of the same paper. These two forms are essentially made up of a trefoil knot and a loop intersecting it.

4. The following knot, which belongs to a species found with every possible number of crossings from 3 upwards. This species furnishes the unique knots with 3 and with 4 crossings, and one of the only two kinds possible with 5.

Its symbol is always of the form

$$\left.\begin{matrix} 2r^3 + (n-3)r^2 \\ 2l^{n-1} + l^2 \end{matrix}\right\} \text{ or more fully } \overset{\diagup\ 2\ \diagdown}{(n-1) \equiv (n-1)},$$

there being n—2 lines in the lower group.

The three last forms have each essentially only one degree of beknottedness. In certain cases (see the foot note *ante* p. 296) we may give two degrees of beknottedness by altering some of the signs —but the knot has then one nugatory intersection, and falls into the class with five crossings.

A number of curious problems are suggested by the process which I employed in the investigation of these six-crossing forms. I give the following as an instance.

Take any arrangement whatever of the first n letters:—Say, for instance,

CNDA . . . LE.

change each to the next in (cyclical order, so that A becomes B, B becomes C, ... , N becomes A) and bring the last of the row to the beginning. The result is

FDAEB . . . . M.

After performing this operation n times we obviously get back the arrangement from which we started. [Thus in seeking all the different forms of knots of a given number of crossings, *one alone* of this set of n need be kept.] The problem is to find sets such that the original combination is repeated after m operations like that above. It is obvious that if m is to be less than n it must be an aliquot part of it, and thus n must be a composite number.

[*April* 11.—The references to Listing's type-symbol here given must be taken in connection with the extracts from his letter, *ante*, p. 316.]

3. Laboratory Notes. By Professor Tait.

(*a.*) Measurement of the Potential, required to produce Sparks of various lengths, in Air at different pressures, by a Holtz machine. By Messrs Macfarlane and Paton.

The general result of these strictly preliminary experiments appears to show that for sparks not exceeding a decimetre in length

(L), taken in air at different pressures (P), between two metal balls of $7^{mm}\cdot 5$ radius, the requisite potential (V), is expressed by the formula

$$V \propto P \sqrt{L}.$$

The Holtz machine employed is a double one, made by Ruhmkorff, and it was used with its small Leyden jars attached. The measurements had to be made with a divided-ring electrometer, so that two insulated balls, at a considerable distance from one another, were connected, one with the machine, the other with the electrometer. With P constant the curve for V closely resembles a parabola between $L=0$ and $L=60^{mm}$, but for higher values of L it appears to tend towards an asymptote at a finite distance from the axis. Thus it would seem that to double the length of very long sparks under these circumstances, a comparatively small percentage increase of potential will suffice. This may enable us to explain some singular peculiarities of lightning. Mr Macfarlane intends to work up this subject very thoroughly, with the help of Thomson's Long Range Electrometer.

(*b.*) The Thermal Conductivity of Gas Coke. By Messrs Knott and Macfarlane.

The method employed was the same as that described (Proc. VIII. 623), but the high conducting power of gas coke for electricity made the experimental work very difficult. The results, so far, are not very consistent with one another, but they appear to point to a diminution of conductivity by rise of temperature.

(*c.*) Preliminary Experiments on the Currents produced by contact of Wires of the same metal at different Temperatures. By Messrs Knott and Macfarlane.

These experiments, so far as they go, confirm the results formerly obtained by Mr Durham, Proc. VII. 788.

(*d.*) On the Relative Percentages of the Atmosphere and of the Ocean which would flow into a given Rent in the Earth's Surface. By Professor Tait.

The note had reference to some sensational statements recently

made under the the title "Are we drying up." But it led also to a curious hydrostatical question as to the equilibrium arrangement of water poured into a shaft already full of air, and supposed to be so deep that in its lower parts the air is compressed to a density exceeding that of water. This suggested numerous questions, such as: What addition to the atmosphere would raise the sea from the earth's surface?

Monday, 5th March 1877.

SIR WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. On the Biliary Secretion with reference to the Action of Cholagogues. By Prof. Rutherford, F.R.SS. L. & E., and M. Vignal.

(*Abstract.*)

Notwithstanding the fact that substances supposed to affect the flow of the bile have been administered to man for more than 2000 years, there is still very great uncertainty as to what does and what does not augment the biliary excretion. The indefinite state of knowledge regarding this point is due to the circumstance that variations in the flow of bile from the human liver are estimated by simply observing the colour of the dejections,—a method that is of necessity exceedingly rough, for it is impossible thus to detect slight variations in the excretion of bile, especially when, as in the case of rhubarb, the substance gives to the dejections a colour similar to that of the bile, and where, as in the case of sodium sulphate, the substance gives rise to copious dejections of a watery character, whereby the colour is diluted.

Even in the case of those substances generally believed to augment the biliary flow, *e.g.*, podophylline, observations on man have entirely failed to determine whether they merely occasion an expulsion of bile from the gall bladder, or an increased secretion by the liver. The determination of the point is of great importance in scientific medicine, for it is calculated to advance our

knowledge of the nature of the diseased condition relieved or cured by the cholagogue.

By experiments on animals, both of the above-mentioned difficulties may be overcome, and definite knowledge arrived at.

The want of precise knowledge in this department of therapeutics induced Nasse, Kölliker, and Müller, Scott, Hughes Bennett, and Röhrig to perform some experiments; but the results have been limited and unsatisfactory, owing to the faultiness of the experimental methods adopted.

In the present research the experiments have been performed on dogs, fasting and curarised, in order that the secretion of bile might be rendered constant. The bile was continuously collected from the common bile duct, and measured every fifteen minutes, and the flow of bile into or out of the gall bladder was eliminated by clamping the cystic duct. The results obtained are as follows:—

1. In a curarised dog that has fasted 18 hours, the secretion of bile is tolerably uniform during the first four or five hours after the commencement of the experiment, but falls slightly as a longer period elapses. Its composition remains constant.

2. Croton oil is a cholagogue of feeble power. The high place assigned to it by Röhrig was probably the result of his imperfect method of experiment.

3. Podophylline is a very powerful stimulant of the liver. During the increased secretion of bile, the percentage amount of the special bile solids is not diminished. If the dose be too large, the secretion of bile is not increased. It is a powerful intestinal irritant.

4. Aloes is a powerful hepatic stimulant. It renders the bile more watery, but at the same time increases the excretion of biliary matter by the liver.

5. Rhubarb is a certain though not a powerful hepatic stimulant. The bile secreted under its influence has the normal composition.

6. Senna is a hepatic stimulant of very feeble power. It renders the bile more watery.

7. Colchicum increases to a considerable extent the amount of biliary matter excreted by the liver, although it renders the bile more watery.

8. Taraxacum is a very feeble hepatic stimulant.

9. Scammony is a very feeble hepatic stimulant.

10. "Euonymin," an impure resinous matter prepared from the bark of *Euonymus atropurpureus*, is a powerful hepatic stimulant. It is not nearly so powerful an irritant of the intestine as podophylline.

11. "Sanguinarin," an impure resinous matter prepared from the *Sanguinaria canadensis*, is a powerful hepatic stimulant. It also stimulates the intestine, but not nearly so powerfully as podophylline.

12. "Iridin," an impure resinous matter prepared from the root of *Iris versicolor*, is a powerful hepatic stimulant. It also stimulates the intestine, but not so powerfully as podophylline.

13. Leptandria is a hepatic stimulant of moderate power. It is a feeble intestinal irritant.

14. Ipecacuan is a powerful hepatic stimulant. It increases slightly the secretion of intestinal mucus; but has no other apparent stimulant effect on the intestine. The bile secreted under the influence of ipecacuan has the normal composition.

15. Colocynth is a powerful hepatic as well as intestinal stimulant. It renders the bile more watery, but increases the secretion of biliary matter.

16. Jalap is a powerful hepatic as well as intestinal stimulant.

17. Sodium sulphate is a hepatic stimulant of considerable power. It also stimulates the intestinal glands.

18. Magnesium sulphate is an intestinal but not a hepatic stimulant.

19. Potassium sulphate is a hepatic and intestinal stimulant of considerable power. Its action on the liver is, however, uncertain, probably owing to its sparing solubility.

20. Sodium phosphate is a powerful hepatic, and a moderately powerful intestinal stimulant.

21. Rochelle salt is a feeble hepatic, but a powerful intestinal stimulant.

22. Sodium chloride is a very feeble hepatic stimulant.

23. Sodium bicarbonate has scarcely any appreciable effect as a hepatic stimulant.

24. Potassium bicarbonate has scarcely any appreciable effect as a hepatic stimulant.

25. Mercuric chloride is a powerful hepatic stimulant. Its stimulant effect on the intestinal glands is feeble.

26. Calomel is a powerful stimulant of the intestinal glands, but does not increase the secretion of bile.

27. Ammonium chloride has no stimulating effect on the liver of the dog.

28. Castor oil stimulates the intestinal glands, but not the liver.

29. Gamboge stimulates the intestinal glands, but not the liver.

The foregoing results, although adding greatly to our knowledge, are in complete harmony with observations on man in every case, save those of calomel and ammonium chloride; but the want of harmony is probably more apparent than real, for there is no evidence that in man these substances really do excite the hepatic cells. They may possibly occasion merely *expulsion* of bile already secreted, or may act on the liver in other ways. The general conclusions of the research are as follows:—

1. That, as the liver of the dog is affected by medicinal agents in the same sense as the human liver, this animal is suitable for observations that cannot be made on man.

2. The attention of medical men is hereby directed to a number of valuable cholagogues, such as euonymin, sanguinarin, iridin, ipecacuan, sodium phosphate, sodium sulphate, &c., hitherto but little or not at all employed as such, owing to the absence of positive information regarding their actions.

3. As regards the whole question of hepatic pharmacology, definite is hereby substituted for indefinite knowledge; for it is not only shown what substances really do act as cholagogues, but it is proved that they all, excepting calomel and ammonium chloride, really do stimulate the liver to secrete more bile.

4. It is shown that, as such substances as magnesium sulphate, gamboge, and castor oil do not increase the secretion of bile, although they irritate the mucous membrane of the duodenum and the remainder of the intestine, the action of stimulants on the secreting apparatus of the liver is probably not reflex from the intestinal mucous membrane, but a direct action either upon the hepatic cells or upon their nerves.

5. It is shown that when a purgative substance is not a cholagogue it diminishes the biliary secretion. The importance of a knowledge of this fact is indicated.

2. Specimen of Auriferous Quartz. By Patrick Dudgeon.

The specimen was found near Wanlockhead, Dumfriesshire, in 1872, by Andrew Gemmell, miner, who unfortunately broke up the piece, and disposed of the fragments to different parties in the locality. Mr Dudgeon was enabled to get possession of all the fragments, and has restored the mass to its original form. The specimen is a very interesting one, being the largest known piece of auriferous quartz found in Scotland; and Mr Dudgeon thought it would interest the fellows of the Society to exhibit it at the meeting before placing it in the Museum of Science and Art, which has now been done with the consent of the different owners.

The following Gentlemen were elected Fellows of the Society:—

George Broadrick, C.E., Claremont Cottage, Leith.
John Napier, Engineer, Lancefield House, Glasgow.
James King, of Campsie, 12 Claremont Terrace, Glasgow.
George Cunningham, C.E., 2 Ainslie Place.

Monday, 19th March 1877.

Sir WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. On a Problem of Arrangements. By Professor Cayley.

It is a well-known problem to find for n letters the number of the arrangements in which no letter occupies its original place; and the solution of it is given by the following general theorem:—viz., the number of the arrangements which satisfy any r conditions is

$$(1-1)(1-2)\ldots\ldots(1-r),$$
$$=1-\Sigma(1)+\Sigma(12)\ldots\pm(12\ldots r),$$

where 1 denotes the whole number of arrangements; (1) the number of them which fail in regard to the first condition; (2) the number which fail in regard to the second condition; (12) the number which fail in regard to the first condition, and also in

regard to the second condition; and so on: $\Sigma(1)$ means $(1)+(2) \ldots +(r)$: $\Sigma(12)$ means $(12)+(13)+(2r) \ldots +(r-1, r)$; and so on, up to $(12 \ldots . r)$, which denotes the number failing in regard to each of the r conditions.

Thus, in the special problem, the first condition is that the letter in the first place shall not be *a;* the second condition is that the letter in the second place shall not be b; and so on; and taking $r=n$, we have the known result, No. =

$$\Pi n - \frac{n}{1}\Pi(n-1) + \frac{n \,.\, n-1}{1.2}\Pi\ (n-2). + \ldots \pm \frac{n \,.\, n-1 \ldots 2 \,.\, 1}{1 \,.\, 2 \ldots n},$$

$$= 1.2.3 \ldots n \left\{ 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1 \,.\, 2 \,.\, 3} \ldots \pm \frac{1}{1 \,.\, 2 \,.\, 3 \ldots n} \right\},$$

giving for the several cases

$$\begin{array}{l} n = 2, 3, 4, 5, 6, 7, \ldots \\ \text{No.} = 1, 2, 9, 44, 265, 1854 \ \ldots \end{array}$$

I proceed to consider the following problem, suggested to me by Professor Tait, in connexion with his theory of knots: to find the number of the arrangements of n letters $a\ b\ c \ldots j\ k$, when the letter in the first place is not a or b, the letter in the second place not b or c, ... the letter in the last place not k or a.

Numbering the conditions $1, 2, 3 \ldots n$, according to the places to which they relate, a single condition is called [1]; two conditions are called [2] or [1, 1], according as the numbers are consecutive or non-consecutive: three conditions are called [3], [2, 1] or [1, 1, 1], according as the numbers are all three consecutive, two consecutive and one not consecutive, or all non-consecutive; and so on: the numbers which refer to the conditions being always written in their natural order, and it being understood that they follow each other cyclically, so that 1 is consecutive to n. Thus, $n=6$, the set 126 of conditions is [3], as consisting of 3 consecutive conditions; and similarly 1346 is [2, 2].

Consider a single condition [1], say this is 1; the arrangements which fail in regard to this condition are those which contain in the first place a or b; whichever it be, the other $n-1$ letters may be arranged in any order whatever; and there are thus $2\,\Pi(n-1)$ failing arrangements.

Next for two conditions; these may be [2], say the conditions are 1 and 2, or else [1 , 1] say they are 1 and 3. In the former case the arrangements which fail are those which contain in the first and second places *ab*, *ac*, or *bc*, and for each of these the other $n-2$ letters may be arranged in any order whatever; there are thus $3\,\Pi\,(n-2)$ failing arrangements. In the latter case the failing arrangements have in the first place *a* or *b*, and in the third place *c* or *d*,—viz., the letters in these two places are $a\,.\,c$, $a\,.\,d$, $b\,.\,c$, or $b\,.\,d$, and in each case the other $n-2$ letters may be arranged in any order whatever: the number of failing arrangements is thus $=2\,.\,2\,.\,\Pi\,(n-2)$. And so in general when the conditions are $[\alpha, \beta, \gamma\,..]$, the number of failing arrangements is $=(\alpha+1)(\beta+1)(\gamma+1)\,..\,\Pi(n-\alpha-\beta-\gamma\,..)$. But for $[n]$, that is for the entire system of the n conditions, the number of failing arrangements is (not as by the rule it should be $=n+1$, but) $=2$, —viz., the only arrangements which fail in regard to each of the n conditions are (as is at once seen), $abc\,..\,jk$, and $bc\,...\,jka$.

Changing now the notation so that [1], [2], [1,1], &c., shall denote the *number* of the conditions [1], [2], [1,1], &c., respectively, it is easy to see the form of the general result: if, for greater clearness, we write $n=6$, we have

$$\begin{array}{llll} 1 & -\Sigma(1) & +\Sigma(12) & -\Sigma(123) \\ \text{No.}=720 & -\{([1]=6)2\}120 & +\left\{\begin{array}{l}([2]\;\;=6)3\\+([1,1]=9)2.2\end{array}\right\}24 & -\left\{\begin{array}{l}([3]\;\;\;\;=\;6)4\\+([2,1]\;\;=12)3.2\\+([1,1,1]=\;\;2.2.2\end{array}\right\}6 \end{array}$$

$$\begin{array}{lll} +\Sigma(1234) & -\Sigma(12345) & +(123456) \\ +\left\{\begin{array}{l}([4\;\;]=6)5\\+([3,1]=6)4.2\\+([2,2]=3)3.3\end{array}\right\}2 & -\{([5]=6)6\}1 & +\{([6]=1)2\} \end{array}$$

or, reducing into numbers, this is

$$\text{No.}=720-1440+1296-672+210-36+2, \qquad =80.$$

The formula for the next succeeding case, $n=7$, gives

$$\text{No.}=5040-10080+9240-5040+1764-392+49-2,=579.$$

Those for the preceding cases, $n=3, 4, 5$, respectively are

$$\begin{aligned} \text{No.} &= 6-12+9-2 &&= 1\\ \text{No.} &= 24-48+40-16+2 &&= 2\\ \text{No.} &= 120-240+210-100+25-2 &&= 13\,. \end{aligned}$$

We have in general $[1]=n$, $[2]=n$, $[1, 1]=\frac{1}{2}n(n-3)$; and in the several columns of the formulæ the sums of the numbers thus represented are equal to the coefficients of $(1+1)^2$, thus, $n=6$ as above, the sums are 6, 15, 20, 15, 6, 1. As regards the calculation of the numbers in question, any symbol $[\alpha, \beta, \gamma]$ is a sum of symbols $[\alpha-\alpha'+\beta-\beta'+\gamma-\gamma'\,.\,.]$, where $\alpha'+\beta'+\gamma'\,.\,.$ is any partition of $n-(\alpha+\beta+\gamma\,.\,.)$; read, of the series of numbers $1, 2, 3 \ldots n$, taken in cyclical order beginning with any number, retain α, omit α', retain β, omit β', retain γ, omit γ', . . Thus in particular, $n=6$, $[1, 1]$ is a sum of symbols $[1-3+1-1]$ and $[1-2+1-2]$; it is clear that any such symbol $[\alpha'-\alpha'+\beta-\beta'\,.\,.]$ is $=n$ or a submultiple of n (in particular if n be prime, the symbol is always $=n$): and we thus in every case obtain the value of $[\alpha, \beta, \gamma\,.\,.]$ by taking for the negative numbers the several partitions of $n-(\alpha+\beta+\gamma\,.\,.)$ and for each symbol $[\alpha-\alpha'+\beta-\beta'+\gamma-\gamma'\,.\,.]$, writing its value, $=n$ or a given submultiple of n, as just mentioned. There would, I think, be no use in pursuing the matter further, by seeking to obtain an analytical expression for the symbols $[\alpha, \beta, \gamma\,.\,.]$.

For the actual formation of the required arrangements, it is of course easy, when all the arrangements are written down, to strike out those which do not satisfy the prescribed conditions, and so obtain the system in question. Or introducing the notion of substitutions,* and accordingly considering each arrangement as derived by a substitution from the primitive arrangement *abcd . . . jk*, we can write down the substitutions which give the system of arrangements in which no letter occupies its original place: viz., we must for this purpose partition the n letters into parts, no part less than 2, and then in each set taking one letter (say the first in alphabetical order) as fixed, permute in every possible way the

* In explanation of the notation of substitutions, observe that (*abcde*) means that *a* is to be changed into *b*, *b* into *c*, *c* into *d*, *d* into *e*, *e* into *a*; and similarly (*ab*)(*cde*) means that *a* is to be changed into *b*, *b* into *a*, *c* into *d*, *d* into *e*, *e* into *c*.

other letters of the set; we thus obtain all the substitutions which move every letter. Thus $n=5$, we obtain the 44 substitutions for the letters *abcde*, viz., these are

(*abcde*), &c., 24 symbols obtained by permuting in every way the four letters *b*, *c*, *d*, *e*.

(*ab*)(*cde*), &c., 20 symbols corresponding to the 10 partitions *ab*, *cde*, and for each of them 2 arrangements such as *cde*, *ced*.

And then if we reject those symbols which contain in any ( ) two consecutive letters, we have the substitutions which give the arrangements wherein the letter in the first place is not *a* or *b*, that in the second place not *b* or *c*, and so on. In particular $n=5$, rejecting the substitutions which contain in any ( ), *ab*, *bc*, *cd*, *de*, or *ea*, we have 13 substitutions, which may be thus arranged:—

(*acbed*), (*ac*)(*bed*), (*acebd*), (*adbec*), (*aedbc*),
(*aedbc*), (*bd*)(*aec*),
(*acedb*), (*ce*)(*adb*),
(*aecbd*), (*ad*)(*bec*),
(*adceb*), (*be*)(*adc*).

Here in the first column performing on the symbol (*acbed*) the substitution (*abcde*), we obtain (*bdcae*), $=$(*aebdc*), the second symbol; and so again and again operating with (*abcde*) we obtain the remaining symbols of the column; these are for this reason said to be of the same type. In like manner symbols of the second column are of the same type; but the symbols in the remaining three columns are each of them a type by itself; viz., operating with (*abcde*) upon (*acebd*) we obtain (*bdace*), $=$(*acebd*); and the like as regards (*adbec*) and (*aedbc*) respectively. The 13 substitutions are thus of 5 different types, or say the arrangements to which they belong, viz.,

cebad, *ceabd*, *cdeab*, *deabc*, *eabcd*,
edacb, *edabc*,
caebd, *daebc*,
edbac, *debac*,
daecb, *deacb*.

are of 5 different types. The question to determine for any value of n, the number of the different types, is, it would appear, a difficult one, and I do not at present enter upon it.

2. On the Construction of the Canon of Sines, for the Decimal Division of the Quadrant. Edward Sang, Esq.

The convenience of having only one system of numeration is so well recognised, that there is no need for any discussion. Already the numerical nomenclatures of all nations having any culture have been converted to one, namely, the decimal system, and traces of the ancient use of dozens, scores, fifteens, or sixties, can be found in only a very few of them. Although we count our hours in sixty minutes, we do not date the present as the year *thirty-one sixties and seventeen.* Yet, in the matters of measure, weight, and value, the old and irksome divisions continue to be used; nay, even in those departments of science which most need laborious calculation, we continue to employ the ancient scale of division.

It is, indeed, remarkable that, while men of business are striving for uniformity in the modes of counting and of measuring, trigonometers and astronomers should remain unconcerned as to the subdivision of arcs and of time. We are rapidly approaching the anomalous position of using the decimal division of the earth's quadrant as the source of our standards of weights and measures, and of yet rejecting that division in our notation of angles.

The want of trigonometrical tables suitable to the new division is the real cause of this backwardness; the construction of these tables is essentially the first step to the universal employment of the decimal system. This has been long and well recognised. In the end of the last century, Borda computed the decimal canon; this computation was superseded by that which the French Government caused to be made under the superintendence of Prony; but neither of these has been put to press. The only centesimal table available to the trigonometer is that given for each minute of the quadrant, in Callet's "Tables Portatives;" it was collated with the manuscripts of Borda and of Prony, but is presented in a most inconvenient form.

The eminent astronomer, Laplace, adopted the decimal division of arcs, of distance and of time, in his "Mécanique Celeste," published in the last year of the century. The force of this illustrious example might long since have gained universal accept-

ance for the system, had not the non-publication of the requisite canon prevented all progress in this direction.

Notwithstanding many solicitations, and even the offer by the English Government to defray a share of the expense, the tables computed in the Bureau du Cadastre remain unpublished; and the fact of their existence remains a discouragement to other computers.

Now, fifty years ago, having fallen upon a method of approximating very rapidly to the roots of numerical equations, published in 1829 under the title, "Solution of Algebraic Equations of all Orders," I applied it to the quinquisection of an arc, and thus obtained directly the sine of any proposed decimal division of the quadrant. After proceeding a short way in the construction of the canon by this method, I laid it aside, from the conviction that the labour could, at best, only produce a repetition of what had already been accomplished.

Many years ago, after having felt for long the want of a table of logarithms more extensive than any existing, I designed a nine-place table up to one million; and having carried the manuscript fifteen-place table up to 300 000, laid it before this Society, whose Council did me the exceedingly great favour of presenting to Government a memorial soliciting aid in the completion and publication.

One of our scientific periodicals, in noticing this memorial, supplied to Government a most potent reason for not acceding to the request, in this, that the labours of Prony in the Bureau du Cadastre had already anticipated and surpassed all that can be done in future in this department of calculation.

Forced thus into a critical examination of the Cadastre Tables, so far as that could be accomplished by help of published documents, I arrived at most unexpected conclusions.

In the first place, the fundamental table, that of the Logarithms of the Prime Numbers, as given by Legendre in his work on Elliptic Functions, was found to be correct up to 2000; that is, as far as Abraham Sharp and other ancient computers had gone. But of the primes between 2000 and 10 000 computed in the Bureau, only five have their logarithms correctly given, while almost all of the other logarthims err on one side.

In the second place, Henry Brigg's original table had been collated by Prony's assistants, and errors in the tenth and higher places had been found; yet no notice had been taken of the very many errors in the fourteenth and even in the thirteenth place.

Thirdly, the system of computation adopted was so imperfect that, although differences of the sixth order were extended to the thirty-sixth place, the results were liable to errors up to the twentieth figure.

Lastly; the Cadastre calculations were used by M. Lefort for correcting Adrian Vlacq's ten-place table (that table of which all the subsequently published seven-place tables are abridgments); with the result of sometimes putting Vlacq in the wrong when he is right. That is to say, the Cadastre calculations cannot be trusted for the compilation of a ten-place logarithmic table.

Such being the case with that part of Prony's great work which was comparable with previously published tables, we are unable to place confidence in the trigonometrical portion, which necessarily is almost entirely new; and we are forced, when contemplating the compilation of the Canon of Sines, to hold the Cadastre Tables as non-existent, or at best, as useful for controlling palpable errors of the press.

In actual trigonometrical calculations we very seldom use the sines and tangents themselves, but employ their logarithms instead; wherefore, both the Canon of Sines and the Logarithmic Canon are needed as the joint foundation of our working tables. Having carried the table of logarithms as far as to 370 000, and being satisfied of the insufficiency of the work done in the Bureau du Cadastre, I resumed the computation of the sines, and have now proceeded to such a length as to be able to submit the methods employed to the Society, and through it to the mathematical public.

The decimal division of the quadrant is effected by bisections and quinquisections, and the first thing to be determined is the order in which these operations should be taken. Now, we obtain the sine of the half arc, not from the sine of the whole arc, but from its cosine, or rather from its co-versed-sine; when the arc is small the co-versed-sine, or defect of the cosine from the radius, is represented by a very small decimal fraction, the number of whose

effective figures is less than the total number of figures in the calculations; we have to extract the square root of its half, which square root, can only be true to as many effective figures, and thus cannot be extended to the full number. If, then, we desire to obtain accuracy to a specified number of decimal places, we must extend our first calculations far beyond. Whereas the sine of an odd sub-multiple is deduced directly from the sine of the whole arc, and the computation is such as to retain the full number of effective figures. Hence we must proceed by first making all the requisite bisections, and thereafter the quinquisections.

There is, however, an obvious exception. The sines of 20^c, 40^c, 60^c and 80^c are obtained by the solution of quadratic equations, so that this quinquisection naturally comes first; the bisections afterwards, and then the remaining quinquisections.

Having prescribed unit in the thirtieth decimal place as the limit of inexactitude, and taken three figures more to obviate the accumulation of the minute errors inevitable in all approximate work, I computed the sines of these four arcs to thirty-three places.

The bisection of the quadrant gave the sine of 50^c; that of 60^c gave for 30^c and for its complement 70^c; that of 20^c gave for 10^c and 90^c, thus completing the table for each tenth part of the quadrant.

In the same way the bisections of 10^c, 30^c, 50^c, 70^c, and 90^c gave the sines of the intermediate fifth degrees.

The bisections were continued in this way until the quadrant was divided into 80 equal parts.

In performing this work, each of the roots

$$\sin a = \sqrt{\{\tfrac{1}{2} - \tfrac{1}{2}\cos 2a\}}, \qquad \cos a = \sqrt{\{\tfrac{1}{2} + \tfrac{1}{2}\cos 2a\}}$$

was extracted twice, and sin a was also found from the division

$$\sin a = \frac{\sin 2a}{2\cos a},$$

by which means the values of $\sin a$ and $\cos a$ were securely got, and the previous computations of $\sin 2a$, $\cos 2a$ again verified. There were, as a matter of course, small errors in the last places, but the results, to the thirtieth place, were made sure.

The quinquisections were obtained by help of the equation

$$16 \sin a^5 - 20 \sin a^3 + 5 \sin a - \sin 5a = 0 = \mathrm{E}\,.$$

Regarding E as a function of $\sin a$, its successive derivatives are

$$\begin{aligned}
{}_1\mathrm{E} &= 80.\sin a^4 - 60.\sin a^2 + 5\\
{}_2\mathrm{E} &= 320.\sin a^3 - 120.\sin a\\
{}_3\mathrm{E} &= 960.\sin a^2 - 120\\
{}_4\mathrm{E} &= 1920.\sin a\\
{}_5\mathrm{E} &= 1920\quad;
\end{aligned}$$

now, while resolving the equation, we get the values of all the derivatives; so, taking advantage of these, we have

$$\cos a^2 = \frac{7}{8} - \frac{1}{960}\,{}_3\mathrm{E}$$

$$\cos 2a = \frac{3}{4} - \frac{1}{480}\,{}_3\mathrm{E}\,.$$

$$\sin 3a = \frac{3}{2}\quad \sin a - \frac{1}{80}\,{}_2\mathrm{E}$$

$$\cos 4a = \frac{1}{4} - \frac{1}{480}\,{}_3\mathrm{E} + \frac{1}{10}\,{}_1\mathrm{E}\,;$$

the first of these gives us, by extracting the square root, cos a.

Applying this method to the arc $1^c\ 25'$, the eightieth part of the quadrant, I obtained the sine and cosine of $25'$, and proceeded to compute the functions of its multiples by help of second differences, according to the well-known formula.

$$\sin(n+1)\,a - 2\sin na + \sin(a-1)\,a = \sin n\,a.2\ \mathrm{ver}\,a\,;$$

and, because the multiplier 2 ver$\,a$ is to be repeatedly used, a table of its multiples was constructed, in the case of $a = 25'$, up to the hundredth, in the cases of $a = 5'$ and $a = 1'$, up to the thousandth multiple.

The sines of these multiples being computed continuously, an error in any part of the work, propagated subsequent errors, which could not possibly be overlooked in comparing the results at each

fifth step with the previously computed test values. In this way immunity from error was obtained, excepting in the last places, where small errors are inevitable.

In performing the multiplication sin na.2 ver a stopping at the thirty-third place, the last figure of each partial product may err in excess or in defect by $\frac{1}{2}$; now it is possible, though not likely, that all of these errors may be on one side, and therefore there is a *possible* error in the last place of the total product, of half as many units as there are lines in the multiplication. By using the table of multiples up to one thousand, we reduce the number of lines to one third, and therefore the possible amount of residual error in the same ratio; so that the auxiliary table both saves labour and augments the exactitude of the result.

These final-place errors were corrected at each fifth step by altering the last figures of the second differences, and thus the accumulation of those errors was prevented. In the whole calculation of the sines of the quarter degrees, it was not found necessary to alter any one second difference to so much as the limit of possible error, and therefore we may hold that the manuscript table of the sines of these arcs is absolutely correct to within the prescribed degree of precision, namely, unit in the thirtieth decimal place.

The next quinquisection, conducted in the same way, gave the sine and cosine of 5′; the functions of whose multiples were obtained as before, and compared, at each fifth step, with the previous work. The table of sines to every fifth minute is already well advanced.

The third quinquisection gave the sine and cosine of the single minute of the decimal division. A table of the multiples of 2 ver 1′ has been constructed up to 1000, and has been used in forming a good beginning of the canon of sines to each single minute.

For the purpose of preventing all error in the record of these calculations, the second differences only were copied from the duplicate scroll calculations, and the successive first differences and sines were thence recomputed on the record sheet. Since any error in copying, or even in the original computation, was necessarily continued and extended into the after part of the record, its detection was rendered certain, so that the recorded results may be implicitly relied on. To make the record more secure, each page

was copied on thin transfer paper, on which no alteration can be made without being obvious.

The method of computing might have been extended to differences of the fourth order, in which case the common multiplier, 4 ver a^2, would have been much smaller, and the terms of the products fewer. But, on the other hand, the entries in the record would have been more, and the effects of the residual errors would have been much greater and more troublesome in correction. For the interpolation of each quarter degree, the saving obtained by the use of fourth differences would have been unappreciable; for those of each fifth and of each single minute, it would have been hardly such as to compensate for the inconveniences just mentioned; but for the future interpolations of each tenth second and of each single second, the fourth differences may be advantageously used.

When the canon of sines shall have been completed, the computation of the working table, that of logarithmic sines, will be easy, particularly by help of my fifteen-place table of logarithms.

Beginning, as John Nepair did, at the sine of the whole quadrant and proceeding downwards, the computed logarithmic sines may be verified by their differences, which are small. When the work has been brought down to the log sin of the half-quadrant, the farther progress is easy and rapid, for the formula

$$2 \sin a . \cos a = \sin 2\,a$$

gives $\log \sin a = \log \frac{1}{2} + \log \sin 2\,a - \log \cos a$, so that the differences of any order may be got at once from the previously tabulated differences of that order, and, what is most worthy of remark, may be used without the fear of an accumulation of residual error.

The table of logarithmic tangents follows as a matter of course.

3. On the Precautions to be taken in recording and using the Records of Original Computations. Edward Sang, Esq.

The real utility of tables of numerical results is only secured by making them accessible to those computers who may require them;

and the essence of their utility lies in this, that the labour of a single computer saves that of many others.

It is indispensable that those who use the tables be able to rely implicitly on the accuracy of the tabulated numbers, and that they have a ready means of detecting any error should the existence of one be suspected.

I do not mean, at present, to say a word on the mechanical arrangements of setting up the types, of stereotyping, revising the proofs, and printing; these have already often been discussed, but I shall take the matter up at this critical point:—The investigator has in his hands a set of printed or of manuscript tables, which he means to use in his researches, and he wishes to know whether the individual book be or be not to be trusted. His confidence must necessarily be influenced by the history of the book and by its co-relatives. Thus, if it be a stereotype copy of a work in extensive circulation, he may accept the general opinion as to its accuracy; but if the table be one seldom used, such as those which serve as the foundation of working tables, this source of confidence fails him.

The nature of the case may be most clearly seen from an example:—

We propose to extend the logarithmic canon beyond the limits to which it has been already printed; this extension must be founded on the logarithms of the prime numbers; now Abraham Sharp computed, to 61 places, the logarithm of every prime number up to 1097; these were printed in Sherwin's collection, and thence reprinted by Callet in his Tables Portatives; shall we then build our more extensive tables on the computation by Sharp? Sharp was known as a most zealous and careful computer; both Sherwin and Callet would take care that the numbers be correctly copied; yet for all that, we cannot venture to found on Sharp's work because there is an essential omission.

If we were to proceed to compute, by help of these, the logarithms of larger primes, and if, after a lengthened series of operations, we were to find a disagreement, we should be left in doubt as to which of the many logarithms that had been used may be in fault; we should have to recompute such of Sharp's logarithms as might be implicated, while the labour and irksomeness of the search would become intolerable.

In all such calculations we seek to arrive at the result by two independent processes. All the use, therefore, that can be made of Sharp's tables, is to hold his work as constituting one of these processes; a great use certainly, yet, at best, only half of what it might have been.

Now, in the computation, to twenty-eight places, of the logarithms of the prime numbers, no error whatever was discovered among those given by Callet; so here we have an instance of records, in themselves quite exact, and yet insufficient to obviate subsequent recomputation.

The means of readily verifying the record are awanting; these means must necessarily vary with the nature of the tabulated functions.

In the volumes containing the computation of the logarithms to twenty-eight places of all primes below ten thousand, which was laid before the Society, the articulate steps of every calculation are recorded and indexed, so that if an error be suspected in any one logarithm, we have the means of instantly verifying the table, or of detecting the source of the error. Had such a record accompanied Sharp's admirable table, the need for subsequent re-calculation would have been entirely obviated.

The vast majority of tables have their arguments arranged with equal differences, consequently the functions progress gradually; and, for the most part, these tables have been constructed by help of differences. It is then sufficient to record the differences along with the values of the functions. For the canon of sines I have found it convenient to place the first difference, with the sign +, and then the second difference, with the sign −, below the preceding sine, as shown below:—

Arc.	*Sine.*
1° 12′	.01759 20113 41099 72108 93607 04490
	+ 15 70551 06706 95090 77046 38118
	− 4 37940 65992 07711 07444
1...13	.01774 90664 47806 67199 70653 42608
	+ 15 70546 68766 29098 69335 30674
	− 4 41815 82853 79716 53726
1...14	.01790 61211 16572 96298 39988 73281

This arrangement enables the computer to examine any sin which he wishes to extract, so as to guard against any typographical error; and, if the table of the multiples of 2 ver 1′ were appended, to check readily the computation itself. When, however, the differences have only a few figures, the ordinary method, of placing them in separate columns, is to be preferred; it saves room, while the practised calculator has no difficulty in performing the requisite additions or subtractions. The utility of this arrangement has been long recognised.

In the case of tables for common use, which are, in general, abbreviations of original calculations to a greater number of places, it is enough to give the first differences when these vary much.

When such means of verification have been provided, the user of the table can make sure that the number which he extracts contains no error; and if all users were to habitually make this examination, printed tables, and above all those printed from stereotype, would be gradually freed from errors of press.

4. On an Unnamed Palæozoic Annelid. By Professor Duns. (Plate IV.)

Cymaderma* (nov. gen.).—*Generis Characteres:*—Corpus cylindricum, elongatum, nudum, striatum; striæ tenues, in ordinem undulatæ, et ubique corpus cingentes, ita ut cutis subrugosa videatur; linea dorsualis continua, alternasque cavaturas ovatus et vertices ostendens; nulla verorum articulamentorum indicia; vestigium tortuosum, bisulcum.

Cymaderma (new genus).—*Generic Characters:*—Body cylindrical, elongated, naked, striated; striæ minute, waved symmetrically, encompassing the form in all parts, and giving to the skin a subrugose appearance; a continuous dorsal line, showing alternate oval depressions and slight ridges; no traces of true articulations; track, tortuous bisulcate.

This annuloid fossil was obtained from upper Carboniferous strata, in a cutting on the Midland Railway, valley of the Ribble, near Settle, Yorkshire. Peculiarities characteristic of the striation and the median dorsal line led me to conclude that the form was probably rare, or one that had not been described. It was thus of importance to get the other part of the slab on which the impres-

* Etymol. κῦμα, *unda*, δέρμα, *cutis*.

sion, or *intaglio*, corresponding to the rounded relief, might be expected. The friend who had forwarded the fossil made diligent search, but found the upper part of the slab had been destroyed. In answer to a request for any animal tracks, or traces of organisms that might be met with in the same, or closely related deposits, he sent me several slabs of compact dark-coloured micaceous, sandy shale, on which some markings occurred. On splitting these, they were found crowded with tracks and casts of forms closely resembling that first sent, though the matrix is lithologically different. These shales intervene between bands of the so-called gritstone, in which the form now noticed was found. Its position was about 16 feet from the surface, in a gritstone bed 4 feet thick, having above a deposit of alternating shales and gritstones nearly 15 feet thick and below a bed of limestone.

The dark-coloured slabs present features of much interest. They contain traces of three species, the prevailing one being identical with that now before us, though none of the characteristic marks are so sharply outlined. They make it clear, however, that the surface on which the median line occurs is dorsal, and they establish the bisulcate character of the track. On the reverse of one slab, the *intaglio* I was anxious to obtain shows distinctly that the dorsal line is ridged. This is also very well marked in another. In both the slight dorsal ridge is represented by a corresponding depression or furrow in the overlying shale. Even the outlines of these tracks are suggestive to the ichnologist. Some of them are comparatively deep, some shallow; in some the bisulcate character is well seen, while in others the lower surface of the track is flat. The explanation of this is obvious. If, for example, we look at the tracks, say, of our common whelks (*Littorinæ*), we see that those formed in shallow shore-pools are flat; those on very wet sand are slightly hollow, their edges presenting a corrugated appearance; while those made on fine sand beginning to dry, are marked at regular intervals with distinct transverse lines. It is not unusual to observe all these crossing one another, or seeming to form loops with each other. The waved whelk (*Buccinum undatem*) makes a bisulcate track on the sand when it is partially dry. Occasionally, however, it trails the sharp edge of its *operculum* in such a way as to give three *sulci*. Ichnologists may thus have the track of only

one form before them, when they describe three or four apparently different ones, and trace them to different species. Thus, such terms as *unisulcus*, *bisulcus*, *trisulcus*, *unisulcus corrugatus*, and the like, instead of indicating so many distinct species, may, in reality, express the varying form of the track of one only. Perhaps, even the Hitchcocks' in their magnificent works on ichnology have not given due weight to this consideration.*

The slabs laid on the table have been examined with great care, in the hope of detecting traces of hairs, setæ, or any other characteristic marks of true recent annelids. But neither hairs nor bristles have been found. On most of the slabs, however, markings occur, which I am inclined to regard as the outlines of organs analogous to the gill-leaves of such forms as *Phyllodoce*. These are worthy the attention of naturalists. They are numerous, and have not, I think, been observed before. While occurring on the same surface as the outlines and tracks of the organisms, only in one instance they seem attached to them, but even in this case the association is doubtful. See Plate IV. fig. 2, for an outline of some of these markings. In one case, in which a good duplicate was obtained, distinct traces as of a fringe appear. An enlarged rough outline of part of this fringe is shown on the Plate, IV. fig. 4. On the same specimen a good many annulate pointed objects occur, two of which are represented at *a* and *b* fig. 2.

There can be no doubt that both tube inhabiting and errant Annelida existed in palæozoic time—traces of some being found even in the Laurentian rocks—though little is known of their true nature and relations. The genera *conchiolites*, *cornulites*, *serpulites*, *spirorbis*, and *trachyderma*, may be said to complete the list of the former, and the genera *arenicolites*, *crossopodia*, *myrianites*, and *nereites* that of the latter. But even some of these generic designations are open to criticism, or cumbered with doubt, and it is questionable if, in any of these cases, we have a true representation of the animal itself. Such considerations add to the value of the specimen now under notice. It has no resemblance to any of the forms just named. The first examples on record having some

* "Ichnology of Massachusetts." By Edward Hitchcock, Boston, 1858. "Supplement to the Ichnology of New England." Edited by C. H. Hitchcock, Boston, 1865.

B. N. Peach, del.

McFarlane & Erskine, Lithrs Edinr

likeness to it were discovered by Mr Dixon, of Unthank,* Northumberland, in fine-grained micaceus carboniferous slabs, in 1838, and sent by him to the Newcastle Museum. In 1844, Dr Emmons,† of Albany, U.S., published an account of annelids found by him in Lower Silurian strata, but none of his figures bear the least resemblance to this. An account is given in the first volume of "The Naturalist" of corresponding markings obtained by Mr Ed. Wood, in 1850, from the Northumberland strata, specimens of which were sent to the Museum, Jermyn Street, which Edward Forbes marked "Casts of Annelid Tracks." M'Coy, who gives in his "British Palæontology"‡ careful descriptions of the genera already named, does seem to have had his attention turned to these. In 1858, Mr Albany Hancock contributed an able paper to the Annals of Natural History, entitled "Remarks on certain Vermiform Fossils found in the Mountain Limestone Districts of the North of England."§ This paper is illustrated by six well-executed plates, the last two of which (xviii. and xix.) bear on the present inquiry. The other figures are undoubtedly mere tracks, and Mr Hancock believes them to be those of crustaceans, but the grounds for this belief are far from clear or satisfactory. Referring to a species of Amphiopoda—*Sulcator arenarius*—he says,—"While forming its track, the animal is never seen; it moves along a little below the surface of the sand, which it pushes upwards with its back, and the arch or tunnel thus formed partially subsides as the creature passes forward, and breaking along the centre the median groove is produced." Now all that this observation shows is, that a median groove is formed in the line of this crustacean's track, though the probability of the realisation of a form like this in such a tunnel cannot be admitted. It would imply that the tunnel

* Since this paper was written I have received from a Fellow of the Society, P. Dudgeon, Esq. of Cargen, several annulose specimens, one of which, from Upper Carboniferous strata, Haltwhistle, Northumberland, not far from this locality, bears a close resemblance to the forms figured by Mr Hancock, and referred to below.

† "The Taconic System." By Eb. Emmons, Abany, 1844.

‡ "Contributions to British Palæontology." By Frederick M'Coy, F.G.S., Cambridge: Macmillan & Co., 1854.

§ Annals and Magazine of Natural History, vol. ii. December 1858. (Plates xviii. and xix.)

in wet sand was kept open till, by gentle infiltration, the hollow was filled, and an exact likeness of the interior of the tunnel realised on the infiltrated matter! A very good example of the forms figured by Mr Hancock occurs on one of the slabs on the table.

At a meeting of the Dublin Geological Society in 1858, Professor Haughton, the president, read a paper "On the Occurrence of some New and Rare Forms of Annelidoid Tracks in the Coal Measures, Lugacurren, Queen's County."* A year later, and again in 1860, he returned to the subject, indicating his belief that the Lugacurren tracks resembled those described by Mr Hancock, and ultimately accepting his theory of their crustacean origin. As Professor Haughton's papers were illustrated by lithographic plates *ad naturam*, they, like Mr Hancock's, are available for comparison with the specimen before us. The Lugacurren specimens differ from the Northumberland forms in having at one part four, in one case, and in another five circular depressions in the median line. They differ from that on the table in having circular depressions confined to a small part of the form, and a median groove, instead of, as here, oval depressions running the whole length of the body, and a median ridge. In neither the English nor the Irish specimens is the characteristic striation so conspicuous as in this. These, however, seem all to be specimens of closely related species. Both Mr Hancock and Professor Haughton think that the depressions in the median groove may have been made by the pygidium of a carboniferous trilobite, the print of the tail being the only trace the animal has left of its having had a place in this deposit! Now such marks as those on the Lugacurren specimens could only have been made by the pygidium being set down vertically, then lifted for a time, and so placed at regular intervals, their being no evidence of dragging,—most unlikely, if not impossible conditions. Fortunately, a good representation of the form figured by Professor Haughton occurs also on these slabs. From this it is evident that the characteristic circular punctures run the whole length of the body.

It seems to me that the reference to the track of *Sulcator arenarius* has been misleading. It can, indeed, have no bearing on any

* Journal of the Geological Society of Dublin, vol. viii. 1857-1860.

of these closely related forms, being unisulcus, destitute of striæ, and not more than three-eighths of an inch wide, while all these are striated and seven-eighths of an inch wide; that under notice being moreover distinctly bisulcate. It is acknowledged that the relations of this to recent forms are obscure. The features which chiefly claim attention are—

First, The outline of the animal.—A glance at the specimen is sufficient to convince us that we have here not a track merely, but the representation of an annulose form. (See Plate IV. fig. 1.) On one of the slabs this is associated with several inches of the track over which it has passed. The median dorsal line is fully exposed above, while the median ridge, which makes the track bisulcate, is precisely what would be formed by the ventral groove of a nereis—*Alitta virens* (Sars), for example. The striæ which pass round the body leave no traces of their outline in this track; but in another, from which the representation of the animal was removed, these striæ are well marked. Again, on the bulged sides of the tortuous outline, the striæ are wider than on the opposite side, while in the comparatively straight parts they are symmetrical. So far as I know, similar markings do not occur on any recent annelid. They are, however, represented, though not so distinctly as here, on a small annelid—*Epitrachys rugosis*—figured by Ehler of Erlangen in his paper on the "Fossil Worms of the Lithographic States of Bavaria."*

Second, The tracks.—They differ widely from the tracks both of *mollusca* and *crustacea*—those of the former being, for the most part, sharp in their turns, and those of the latter consist generally of lines more or less straight, not tortuous. In addition, they have two furrows divided by a distinct median ridge. I am sure that had the able observers named above seen such specimens of the tracks, and also of the *intaglios* of the rounded dorsal surface as are now on the table, they would not have questioned the true annulose character of this organism.

Third, The median dorsal line.—This is exceedingly well represented, not only on the outline of the animal itself, but also in one of the intaglios referred to. It consists of small, shallow, oval

* Ueber fossile Würmer aus dem lithographischen Schiefer in Bayern Cassel, 1869.

depressions, deepest in the centre, and narrowing at each end, where they meet a slight ridge which stretches between the depressions, giving to the line, looked at from a short distance, a chain-like appearance. Were the branchial tufts of some recent annelids plucked out, we would have a somewhat similar appearance.

Fourth, The characteristic striation.—This is most distinctly and even sharply marked on the form in the gritstone slab. It is also, though less definitely, marked on some of the softer micaceous slabs. Mr Hancock says, with reference to his specimen which has most resemblance to this—"The transverse striæ on the surface of the grooved form certainly gives it much the appearance of some organism;" but the value of this acknowledgment is lost by the supposition that the striæ might have been "produced by the intermitting progress of the animal." Now it is simply impossible that such striation as is seen here could have been produced in this manner.

The expression "*ondeleusement et symmétriquement*," used by Cuvier in describing the striæ on the shell of a cephalopode, very well indicates a leading feature of this striation. Indeed, the symmetry of these beautifully regular undulating striæ may be best understood by comparing them with the striæ on the shells both of recent and fossil nautilidæ. Fig 3 is intended for a representation, *ad naturam*, of the characteristic striation, but the striæ are sharper and better defined than shown on the figure.

In conclusion, it will be seen that the distinctive features of the specimen now brought under the notice of the Society are the median dorsal line and the waved striation. In the generic features set down at the head of this paper, I have described the former thus,—linea dorsualis continua, alternasque cavaturas ovatas et vertices ortendens. As, however, uncertainty attaches to the nature of this line, the latter—the striation—may be taken as the outstanding generic feature,—striæ tenues, in ordinem undulatæ, et ubique corpus cingentes, ita ut cutis subrugosa videatur. The term Cymaderma, or wave-skin, is proposed for a genus, whose true zoological position is as yet uncertain. Should the examination of other specimens show that the oval depressions in the median dorsal line are only specific marks, not points of insertion of organs, and the striæ mere lines formed by the contraction of the *cutis*—a most

unlikely circumstance—the organism would have closer nemertean than annelid relations. But, if proof be ultimately obtained that the branchiæ-like organs referred to above were connected with the oval depressions, and that the transverse markings are really not striæ but annuli, the zoological position of the animal will be among true annelids, characterised, however, by structural features widely divergent from recent forms.

5. On Eisenstein's Continued Fraction. By Thomas Muir, M.A.

6. Note on an Infinitude of Operations. By Thomas Muir, M.A.

The assumption that there is a limit in Professor Tait's problem regarding the interpretation of $\mathrm{Lim}_{n=\infty}(\cos^n x)$ means that if we start with an angle x, and find the number which is its cosine, then the number which is the cosine of this number, and so on, we shall at last come to a limiting result ω, such that $\cos\omega = \omega$. The problem is thus transformed into the solution of the equation $\omega = \cos\omega$, which may be accomplished as follows:—

$$\omega = \cos\omega,$$

$$\therefore \qquad \omega > 1 - \frac{\omega}{2},$$

whence $$\omega > \sqrt{3} - 1,$$

$$> \cdot 73205 \ldots .$$

Taking therefore ·733 as a first approximation, we find that with it

$$\omega - \cos\omega = -\cdot 01$$

and taking ·75 or $\frac{3}{4}$ as another approximation, which is readily seen to differ from the former by erring in excess, we find that with it

$$\omega - \omega = \cos \perp \cdot 019 \ldots .$$

from which two results by the regula falsi we derive a better approximation, viz.,

$$\cdot 733 - \frac{\cdot 75 - \cdot 733}{\cdot 019 + \cdot 01} \times \cdot 01$$

i.e., $\cdot 7388 \ldots .$

Continuing in this way, it is found that

$$\operatorname{Lim}_{n=\infty}(\cos_n x) = \cdot 7390852,$$

a result correct to at least the fifth place.

This peculiar constant we should expect to find connected in some way with π, and that such a connection exists is easily seen from the formula—

$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right) \ldots .$$

from which we have

$$\omega = \left(1 - \frac{4\omega^2}{\pi^2}\right)\left(1 - \frac{4\omega^2}{3^2\pi^2}\right)\left(1 - \frac{4\omega^2}{5^2\pi^2}\right) \ldots .$$

An *explicit* expression of the same relation is obtained from the identity

$$\cos^{-1} x = \frac{\pi}{2} - x - \frac{1}{2} \cdot \frac{x^3}{3} - \frac{1 \cdot 3}{2 . 4} \frac{x^5}{5} - \ldots .$$

which gives

$$\omega = \frac{\pi}{2} - \omega - \frac{1}{2} \cdot \frac{\omega^3}{3} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\omega^5}{5} - \ldots .$$

whence $$\pi = 4\omega + \frac{\omega^3}{3} + \frac{3}{4} \cdot \frac{\omega^5}{5} + \ldots .$$

the result of the "reversion" of which is

$$\cos^{\infty} x = \frac{\pi}{4} - \frac{1}{3 . 4}\left(\frac{\pi}{4}\right)^3 - \frac{1}{3 . 4 . 5}\left(\frac{\pi}{4}\right)^5 - \ldots .$$

The corresponding limit in the case of other functions may be similarly interpreted; that is to say, the limit of $\phi^n(x)$, if such be possible, when n is indefinitely increased, is a root of the equation $\phi(x)=x$. We may view such limits as implying an *infinitude of operations;* and we have seen that when this infinitude of operations is carried out upon any value (within certain limits) of the independent variable, the result is always the same; in other words, we have seen that those functions, in the case of which the limit is possible, are *levelling* functions, having the property of bringing everything they act on to the same dead level. In this respect they may be compared with Euler's expression

$$\frac{1}{2}x + \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots .$$

which for all values of x within certain limits equals $\frac{\pi}{2}$; and this suggests the possibility of finding a similar expression for $\cos^{\infty}x$ by means of Lagrange's or Fourier's theorem.

7. Note on Determinant Expressions for the Sum of a Harmonical Progression. By Thomas Muir, M.A.

(Received February 27—Read March 1877.)

Taking the harmonical progression

$$\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \ldots .$$

and denoting the sum of n terms of it by S_n we have by means of Euler's transformation,

$$S_n = \cfrac{1}{a - \cfrac{a^2}{2a+b - \cfrac{(a+b)^2}{2a+3b - \cfrac{(a+2b)^2}{2a+5b - \cdots - \cfrac{\{a+(n-2)b\}^2}{2a+(2n-3)b}}}}}$$

and therefore from the theory of continuants,

$$S^n = \frac{\begin{vmatrix} 2a+b & (a+b)^2 & 0 & 0 & \cdots \\ 1 & 2a+3b & (a+2b)^2 & 0 & \\ 0 & 1 & 2a+5b & (a+3b)^2 & \cdots \\ 0 & 0 & 1 & 2a+7b & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}}{\begin{vmatrix} a & a^2 & 0 & 0 & \cdots \\ 1 & 2a+b & (a+b)^2 & 0 & \\ 0 & 1 & 2a+3b & (a+2b)^2 & \cdots \\ 0 & 0 & 1 & 2a+5b & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}}$$

where the denominator is of the n^{th} order and the numerator of the $(n-1)^{th}$, being formed from the denominator by the omission of the first row and first column.

There is, however, another such expression for S_n of more interest and less likely to occur to one, viz.,

$$\frac{\begin{vmatrix} 1 & -b & -b & -b & -b & \cdots\cdots & -b \\ 1 & na & -2b & -3b & -4b & \cdots\cdots & -(n-1)b \\ 1 & (n-1)a & a & -2b & -3b & \cdots\cdots & -(n-1)b \\ 1 & (n-2)a & a & a & -2b & \cdots\cdots & -(n-1)b \\ 1 & (n-3)a & a & a & a & \cdots\cdots & -(n-1)b \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2a & a & a & a & \cdots\cdots & a \end{vmatrix}}{a(a+b)(a+2b)\cdots\cdots\{a+(n-1)b\}}$$

This is easily verified for the cases where $n=2$ and $n=3$. When $n=4$ we have

$$a(a+b)(a+2b)(a+3b)S_4 = \begin{vmatrix} 1 & -b & -b & -b \\ 1 & 4a & -2b & -3b \\ 1 & 3a & a & -3b \\ 1 & 2a & a & a \end{vmatrix} = \begin{vmatrix} 1 & -b & -b & 2b \\ 1 & 4a & -2b & 0 \\ 1 & 3a & a & 0 \\ 1 & 2a & a & a+3b \end{vmatrix}$$

$$= (a+3b) \begin{vmatrix} 1 & -b & -b \\ 1 & 4a & -2b \\ 1 & 3a & a \end{vmatrix} - 2b \begin{vmatrix} 1 & 4a & -2b \\ 1 & 3a & a \\ 1 & 2a & a \end{vmatrix}$$

$$= (a+3b, \begin{vmatrix} 1 & -b & -b \\ 1 & 3a & -2b \\ 1 & 2a & a \end{vmatrix} + (a+3b) \begin{vmatrix} 1 & 0 & -b \\ 1 & a & -2b \\ 1 & a & a \end{vmatrix} - 2b \begin{vmatrix} 1 & 4a & -2b \\ 1 & 3a & a \\ 1 & 2a & a \end{vmatrix}$$

$$= (a+3b, \begin{vmatrix} 1 & -b & -b \\ 1 & 3a & - \\ 1 & 2a & a \end{vmatrix} + (a+b)a(a+2b)\,,$$

for the last two determinants in the preceding line are each equal to $a(a+2b)$. Thus we have

$$S_4 = \frac{\begin{vmatrix} 1 & -b & -b \\ 1 & 3a & -2b \\ 1 & 2a & a \end{vmatrix}}{a(a+b)(a+2b)} + \frac{1}{a+3b}$$

$$= \quad S_3 \quad + \frac{1}{a+3b}$$

as it should be. And this method of showing that the validity of the fourth case is dependent on that of the third is applicable in other case.

8. Sevenfold Knottiness. By Prof. Tait.

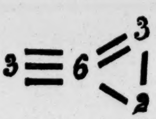
(Abstract.)

From the point of view of the Hypothesis of Vortex Atoms, it becomes a question of great importance to find how many distinct forms there are of knots with a given amount of knottiness. The enormous numbers of lines in the spectra of certain elementary substances show that the form of the corresponding Vortex Atoms cannot be regarded as very simple. But this is no objection against, it is rather an argument in favour of the truth of, the Hypothesis.

For not only are the great majority of possible knots not stable forms for vortices; but altogether independently of the question of kinetic stability, the number of distinct forms with each degree of knottiness is exceedingly small,—very much smaller than I was prepared to find it. I have already stated that for three, four, five, and sixfold knottiness, the numbers are only 1, 1, 2, 4. For a reason given in my first paper, knots whose number of crossings is a multiple of 6 form an exceptional class: so I thought it might be useful to discover and to figure all the distinct forms with seven-fold knottiness. Eight and higher numbers are not likely to be attacked by a rigorous process until the methods are immensely simplified. The method of partitions, supplemented by the graphic formulæ of my last paper, is to some extent tentative. I have verified the present results by means of it, and have extended it to 8-fold knottiness, but I am not certain that I have got *all* the possible forms of the latter.

As I did not see how to abridge the process, I wrote out all the admissible permutations of the seven letters in the even places of the scheme. These I found to be 579, five of which were, of course, unique. The others (as 7 is a prime number) were divisible into 82 groups—those of each group being mutually equivalent. On examination, it was found that only 22 of the 87 selected arrangements satisfied the criterion for possible knots (see I § (*b*) of my paper, *ante* p. 238), and several even of these were repetitious. These repetitions were of two kinds—1st, the mere inversion of the order of the scheme; 2d, the relative positions of a 3-fold and a 4-fold knot which in certain cases were found combined as a 7-fold form. Clearing off these repetitions, and along with them a form really belonging to 6-fold knots (because consisting of two trefoil knots and one nugatory intersection), there remain only *eleven* distinct forms of the 7th order. These are as follows:—

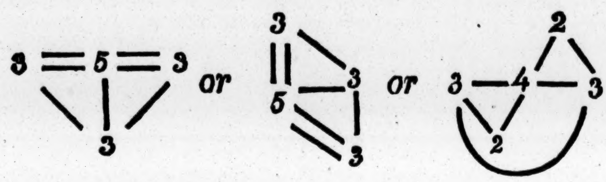
1.



This has a great many forms, with correspondingly different

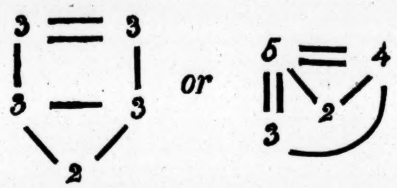
symbols, being a mere compound of a 3-fold and a 4-fold knot, which may have *any* relative positions on the string.

2.

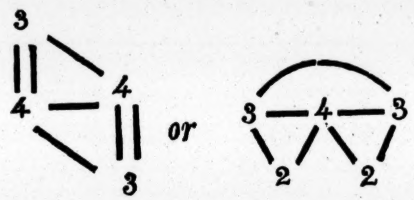


This is one of Listing's knots.

3.

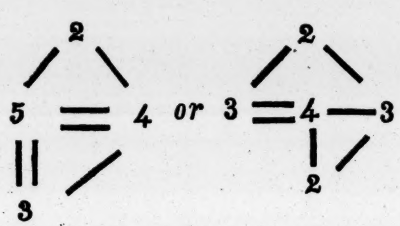


4.



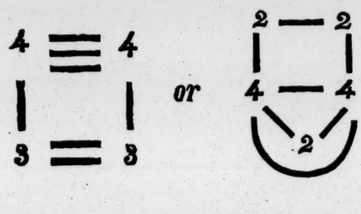
Listing has shown that this is deformable into 2 above.

5.

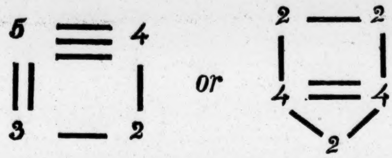


I find that this can be deformed into 3 above. It is figured in my paper on *Links*, *ante*, p. 325, first woodcut.

6.

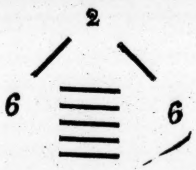


7.



This can be deformed into 6 above

8.

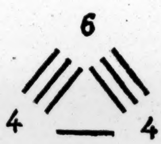


This species of knot occurs for *all* numbers of intersections greater than 2.

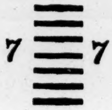
9.

This is the 7 knot which Listing does not sketch. *Ante*, p. 311.

10.



11.



This is the simple twist, which occurs for every *odd* number of intersections.

As 2 and 4, 3 and 5, 6 and 7 are capable of being deformed into one another, three of them are not independent forms, and thus the number of distinct forms of seven-fold knots is only *eight*.

Drawings of various forms of each of these knots were given, as well as indications of the modes in which they can be formed from knottinesses of lower orders.

Monday, 2d April 1877.

SIR WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. Professor GEIKIE exhibited a large Map showing the progress of the Geological Survey of Scotland.

2. Notice of a Saline Water from the Volcanic Rocks of Linlithgow. By Professor Geikie, F.R.S.

From a boring which has recently been made about a mile west from the town of Linlithgow, water has been obtained differing so much in character from that of the usual wells and springs of the district that some notice of it deserves to be placed on record. When the fact was communicated to me I was asked to explain by what means sea-water could obtain access to underground rocks in an inland district. On visiting the ground I found the site of the bore to be among some hollows of the gravel and sand which cover the country between Falkirk and Linlithgow, its height being about 165 feet above the sea, from which it was distant about three miles. The ordinary wells of the district are situated in the superficial deposits, and supply good potable water, though the supply is necessarily limited. A more copious flow being desired the bore was sunk through the sands, gravels, and clays (here rather more than 100 feet thick), and then entered upon a succession of green, brown, blue, and red "whinstone." After a depth of 317 feet had been passed through, consisting entirely of these alternations of "whinstone," a sample of the tolerably copious supply of water which had now appeared was drawn up and sent for analysis to Dr Stevenson Macadam, whose results were as follow:—

Chloride of sodium,	118·76	grains per	imperial gallon.
Sulphate of lime,	7·94	"	"
Chloride of calcium,	6·78	"	"
Chloride of magnesium,	7·83	"	"
Chloride of potassium,	1·46	"	"

Carbonate of lime,	20·23	grains per imperial gallon.	
Carbonate of magnesia,	3·48	"	"
Oxide of iron,	0·34	"	"
Phosphates,	0·71	"	"
Soluble silica,	0·64	"	"
Organic and volatile matters,	2·37	"	"
	170·54		

This large admixture of saline ingredients rendered the boring unavailable for the increase of the water-supply; but the boring-rods were driven 30 feet further in the hope that possibly this mineral water might find some other subterranean means of escape. The hope was of course disappointed, and the operation terminated at a depth of 451 feet from the surface, the lower 348 feet of the boring having been found to pass wholly through "whinstone" in numerous bands of varying hardness and colour. Two samples of the water, taken when the ultimate depth had been reached, were submitted for analysis to different chemists, and gave nearly similar results. The proportions of salts obtained by Mr Robert M'Alley, Falkirk, were the following:—

Carbonate of lime,	7	grains per gallon.	
Carbonate of magnesia,	·60	"	"
Sulphate of lime,	·92	"	"
Chloride of calcium,	1·22	"	"
Common salt,	134·76	"	"
Alumina,	·20	"	,
Siliceous matter,	1·20	"	"
Volatile and organic matter,	1·60	"	"
	147·50		

One character of the water not noticed in the analyses, but distinctly perceptible to me in a freshly drawn sample, was the odour of sulphuretted hydrogen. I may add that the bore was begun from the bottom of a previously made well 18 feet from the surface, and that I found the water flowing out abundantly from the top of the bore-tube into the well from which it was temporarily pumped away.

It was evident that the idea of any subterranean communication

with the sea was quite inadmissible. In the first place, the locality is three miles from the nearest part of the sea-coast, with a ridge of high ground intervening. In the second place, between it and the sea-margin lie the numerous deep pits of the Borrowstounness and Kinneil coal-fields, which, it may be supposed, must prevent at least any superficial water-communication. And in the third place, the proportions of the various salts contained in the water are quite different from those which would have resulted from the mere access of ordinary sea-water. The salts can only have been derived from the subterranean rocks traversed by the water.

An acquaintance with the geology of the district enables me to recognise the various kinds of "whinstone" found in the bore. They are successive beds of the dull green and brown, usually more or less decomposed, dolerites (diabases) forming the long volcanic range between Bathgate and the sea, and which were poured out partly as submarine and partly as subaërial lavas during the deposition of the Carboniferous Limestone, or Lower Coal series of Linlithgowshire. Occasional bands of green and red tuff mark intervals between the lava-flows. Since the water from the superficial gravels was of the usual potable quality there can be no doubt that the saline impregnation comes from the underlying rocks.

So far as I am aware, the first detection of soda as a chemical constituent of rocks was made by Dr Robert Kennedy, and announced to this Society as far back as December 1798. He analysed the specimens of whinstone employed by Sir James Hall in his classical experiments upon the fusibility of whinstone and lava, and found the constant presence of soda to the extent of four or five per cent., with one per cent. of muriatic acid. He further examined pieces of sandstone from the neighbourhood of Edinburgh and elsewhere, with the invariable result of detecting an appreciable quantity of common salt in them. Since the early date of his researches chloride of sodium has been found very widely diffused among the minerals and rocks of the earth's crust.

There seems to be three chief sources from which rocks derive their chloride of sodium:—1. Rain; 2. The evaporated water of old salt lakes and inland seas; 3. Volcanic sublimations.

1. *Rain.*—The researches of Dr Angus Smith * into the compo-

* See his *Air and Rain.*

sition of rain in different parts of the country have shown the very general prevalence of chlorides, and particularly of chloride of sodium, in the air. As might be expected, the proportion of chlorides is greatest nearest to the sea, though abnormally large quantities are found in the air of manufacturing towns as the result of the combustion of coal, &c. On the west side of Britain the proportion of hydrochloric acid in rain was found sometimes to amount to nearly four grains per gallon, or 56 parts in a million. On the east side of Scotland the proportion sinks to ·9 grain per gallon, or 12 parts in the million. There can be little doubt that most of this is chloride of sodium. Dr Smith has pointed out the curious fact that this salt cannot be conveyed into the atmosphere merely in spray driven from the surface of the sea by high winds, for if that were the case the composition of the rain should be approximately like that of sea-water. But the saline ingredients do not occur at all in similar proportions. It seems reasonable to suppose that the superficial parts of rocks liable to be saturated with rain-water must thereby receive an appreciable amount of chloride of sodium. In this case it is evident that much care should be exercised in procuring for analysis portions of rock which lie beyond the reach of this surface saturation. Possibly in some of the instances cited by Kennedy, in the paper already referred to, the common salt may have been introduced by the action of rain.

2. *The Deposit of Salts on the Floor of Old Lakes and Inland Seas.*—This mode of origin is doubtless by much the most important source of the chloride of sodium in rocks. When we consider the large proportion of marine strata in the stratified part of the earth's crust it is surprising, as De la Beche remarked long ago,* that saline waters are not more abundant than they really are. Dr Sterry Hunt has pointed to the mineral waters of Canada and the North-Eastern States as probably deriving their salts from the original sea-water of palæozoic times still imprisoned within the pores of the rocks.† I need not refer to the abundant deposits of rock-salt and gypsum, as well as of saliferous and gypsiferous clays, which occur in so many districts of the world.

3. *Volcanic Sublimations.*—The occurrence of incrustations and

* Researches in Theoretical Geology.

† "Essays in Chemical Geology, 1875."

stalactites of common salt upon the walls and slopes of a volcanic crater after eruption, and the appearance of the same substance upon the surface or within the chinks of recently-consolidated lava streams, have long been observed. On Vesuvius vast quantities of salt have been gathered again and again, and have been used by the inhabitants at the foot of the mountain. Among the Icelandic volcanoes the same fact has occurred. But besides this outward and conspicuous manifestation, chemical analysis has revealed the presence of chloride of sodium in many volcanic rocks, either diffused through their crystalline matrix or entangled within their individual constituent minerals. In such volcanic minerals as Hauyne, Nosean and Sodalite it has been found to be noticeably present. It has been extracted even by pure water from clinkstones, basalts, and from various plutonic rocks.

There can be little doubt that it is to this volcanic origin that the large admixture of salt in the Linlithgow water is to be traced. The bore may have reached a stratum of ancient lava or tuff, originally permeated or incrusted with salt, which may have been dissolved by percolating water, and yet have remained in solution below for want of any ready means by which the water could escape to the surface. This escape is now provided by the bore, and the water is consequently now removing the salt from the rock.

It may be mentioned that some of the best known saline waters of Scotland rise from volcanic rocks. At the Bridge of Allan, water containing 95 grains of chloride of sodium in the gallon takes its rise from near the top of the enormous mass of lavas and tuffs erupted during the Lower Old Red Sandstone period, and forming now the chain of the Ochil Hills. At Pitcaithley, Bridge of Earn, water containing 114 grains of the same salt in the gallon rises from the same volcanic band, while at Dunblane, water with 48 grains of common salt in the gallon comes through the lower parts of the red sandstones of that district, which lie upon the same great volcanic series. In most of these cases the rise of the saline water appears to have been determined by an accident, such as the sinking of a bore for water, or, as at Bridge of Allan, during the search for minerals.

3. On the Arrangement and Relations of the Great Nerve-Cords in the Marine Annelids. By W. C. M'Intosh, M.D., F.R.S.E., F.L.S.

(*Abstract.*)

FAM. EUPHROSYNIDÆ.—In *Euphrosyne foliosa*, Aud. and Ed., the separate nerve-cords are comparatively large, and lie quite within the body-wall, the oblique muscles, which generally bound the longitudinal ventral muscles, decussating beneath them.

AMPHINOMIDÆ.—The cords are somewhat small and flattened in *Chloeia*, and occupy an area bounded internally by a transverse band of fibres, and externally by the circular muscular layer and the hypodermic basement-tissue. The oblique muscles are attached at the outer border of each trunk.

APHRODITIDÆ.—The nerve-cords in *Aphrodita aculeata*, L., occur in a transversely elongated space between the ventral attachments of the oblique muscles, and bounded externally by the hypodermic basement-tissue and the cuticle. In *Hermione hystrix*, Sav., again, they lie—as distinct trunks—in a well-defined hypodermic area within the dense cuticle, and separated by an interval from the attachment of the oblique muscle on each side.

POLYNOIDÆ.—In *Lepidonotus squamatus*, L., the cords occupy a hypodermic area between the ventral longitudinal muscles. The oblique muscles pierce the vertical at the upper and outer angle of the space, and are attached to the hypodermic basement-tissue, external and superior to the cords. A belt of hypoderm intervenes between the latter and the cuticle. In *Polynoë scolopendrina*, Sav., the interval between the ventral attachments of the oblique muscles is less, and the cords are bounded superiorly by a special longitudinal muscle.

ACŒTIDÆ.—In *Panthalis Œrstedi*, Kbg., the trunks are situated in the hypodermic region between the ventral longitudinal muscles, a thin layer of the former tissue occurring between them and the cuticle. The great oblique muscles pass down to their upper and outer border. The space between the ventral longitudinal muscles is less than in the Polynoidæ.

SIGALIONIDÆ.—The space between the ventral longitudinal muscles anteriorly in *Sthenelais boa*, Johnst., is still more narrowed

than in *Panthalis*, and the hypodermic area for the nerve is thus increased in depth. Superiorly the arch is completely covered by the insertions of the vertical and oblique muscles; and it is interesting that the latter do not pierce the former (which occupy the middle line), but are attached to the basement-tissue below them, on each side of the nerve-area. The cords are almond-shaped in transverse section, whereas in *S. Mathildæ*, Aud. and Ed., they are round, and the hypodermic area enclosing them is much more expanded inferiorly.

Nephthydidæ.—In *Nephthys cœca*, Fabr., the combined oblique and vertical are attached along the entire arch of basement-tissue, above the nerve-area. A broad hypodermic belt exists above the nerve trunks, and a narrower between them and the cuticle.

Phyllodocidæ.—The nerves in *Phyllodoce grœnlandica*, Œrst., are situated within the circular muscular coat, and above the insertions of the oblique muscles (which decussate in the middle line), as well as such fibres of the vertical muscles as are inserted into the basement-tissue of the ventral hypoderm. In *Eteone picta*, De Quatref., and *Eulalia viridis*, O. F. Müller certain fibres of the oblique pass at intervals right over the cords, so as to form a continuous band from side to side, and in the former an interval between the oblique muscles is indicated.

In *Alciope* the cords also lie within the circular muscular layer—in the interval between the longitudinal ventral. The oblique pass below the cords, but do not appear to meet in the middle line. The sole specimen, however, is indifferently preserved for microscopic work.

Hesionidæ.—The trunks in *Ophiodromus vittatus*, Sars., have in the anterior region passed below the ventral attachment of the oblique muscles to the basement-tissue. A thin stratum of longitudinal fibres occurs superiorly, while externally is a thickened hypoderm. The nerves seem to be proportionally large.

Syllidæ.—In *Syllis armillaris*, O. F. Müller, the nerve-cords are also comparatively large. A considerable depth of the closely approximated ventral longitudinal muscles shuts them from the very thin hypodermic elements within the thickened cuticle, except in the intervals between the ganglia, where there is a

slender pedicle. The oblique muscles bend below the cords to be attached to the raphe in the same intervals. Fibres from the walls of the alimentary canal also descend to the raphe on each side of the trunks.

NEREIDÆ.—In *Nereis pelagica*, L., the nerve-cords lie rather above the attachments of the oblique muscles to the hypodermic basement-tissue, the area being continued to the hypoderm by a central pedicle. *Nereis* (*Alitta*) *virens*, Sars., an epitocous form, likewise has the oblique muscles attached on each side of the pedicle of the nerve-area, while the vertical are inserted in a chitinous arch at the upper and lateral regions. Two well-marked longitudinal muscles lie over the cords. Several neural canals exist,—viz., two large infero-lateral, a single superior median, and a smaller, a little below the latter, on each side. In *Nereis diversicolor*, O. F. Müller, each cord has a neural canal of considerable size towards its inferior border.

STAUROCEPHALIDÆ.—The cords are large in *Staurocephalus rubrovittatus*, Grube, and occur in the somewhat wide interval between the ventral attachments of the oblique muscles. Externally are the basement-tissue, hypoderm, and cuticle.

LUMBRINEREIDÆ.—In a *Lumbriconereis* from Herm, a large neural canal exists above the cords, which are carried inward by the approximation of the great longitudinal ventral muscles. The oblique meet over the neural canal. Externally are basement-tissue, hypoderm, and cuticle. The nerves are pressed further inward in *Notocirrus tricolor*, Johnst., the oblique muscles being attached to the summit of the nerve-area (laterally), outside the fibres forming the special chamber for the vessel. Some fibres pass down each side, and cross below the nerve-area.

The neural canal in the posterior third of *Lysidice ninetta*, Aud. and Ed., is situated toward the ventral border of the ganglia, but between the cords in the intervals. The oblique muscles pass below the cords. The latter takes place likewise in *Palolo viridis*, Gray.

EUNICIDÆ.—The cords in *Marphysa sanguinea*, Mont., lie between the greatly developed longitudinal ventral muscles, and present the aspect of a margin to the large median neural canal. The fibres of the oblique muscles are attached to the upper and

outer border of the region,—a few fibres passing downward to curve outward at the circular coat of the body-wall. Externally, between the latter coat (and a layer of somewhat isolated longitudinal fibres within it), are the hypoderm and cuticle. In *Eunice norvegica*, L., a similar arrangement occurs in the anterior third, and a large median neural canal lies below the cords. The strong oblique muscles pass down to the ventral hypoderm. Distinct muscular bands also enclose the ventral blood-vessel and nerve trunk in a tunnel. A little behind the middle of *Eunice Harassii*, Aud. and Ed., the cords present the same external coverings, only, from the great size of the oblique and vertical muscles, they are somewhat supported by the latter inferiorly. The same arrangement occurs in a large male *Eunice* from the "Porcupine." The constancy of the muscular tunnel for the nerve-cords and ventral blood-vessel is interesting.

Onuphididæ.—The oblique muscles in both *Nothria conchylega*, Sars., and *Hyalinœcia tubicola*, O. F. Müller, are well developed, and not only meet, but slightly cross, in the middle line. The nerve-cords lie in the angle of decussation superiorly, and have a single neural canal of considerable size towards the lower border. Besides the oblique are externally the circular muscular coat (which, in *H. tubicola*, is specially developed in the median line), a narrow band of hypoderm, and the dense cuticle.

Goniadidæ.—In the anterior region of *Goniada maculata*, Œrst., the powerful oblique muscles sweep from below the bristle-bundles on each side, with a slight inclination downward and inward, and meet for insertion on each side of the hypodermic wedge above the nerves. The latter occupy a somewhat triangular area of the hypoderm, and each has a small neural canal toward the upper and narrow part. In *Eone Nordmanni*, Mgrn., the nerves are proportionally smaller, and the hypodermic area less.

Glyceridæ.—The cords at the anterior third of *Glycera capitata*, Œrst., are large, and occur in a hypodermic region, wedged between the great longitudinal ventral muscles, which touch in the middle line, so as to form an arch over the nerves. The great external circular muscular layer ceases before reaching the nerve-area, so that externally the latter has only the hypoderm and the specially thickened cuticle. The oblique muscles are very slightly

developed in this species, and their terminations over the cords can only be seen occasionally. Each cord has a neural canal at its upper third, near the middle line. *Glycera Gœsi*, Mgrn., shows nearly the same arrangements; while *G. setosa*, Œrst., exhibits more evident separation of the cords. The insertions of the oblique muscles over the nerve-area are best seen in *G. alba*, Rathke, a large species, in which the cords are proportionally less.

Ariciidæ.—In the anterior region of *Aricia latreillii*, Aud. and Ed., the cords are situated at the upper part of a somewhat triangular hypodermic area, bounded laterally by that part of the circular coat clasping the ventral longitudinal muscles. Powerful oblique muscles meet for insertion over each cord. In the same region of *Scolopos armiger*, O. F. Müller, the cords also lie beneath the point of union of the greatly developed and nearly horizontal oblique muscles, and supported laterally by the massive edges of the ventral longitudinal. A single neural canal exists superiorly. Between the cords and the hypoderm is the thick circular muscular coat. In the middle of the body the nerves are thrust upward by the great ventral muscles, which are only separated by the narrow pedicle of the area.

Opheliidæ.—Throughout the greater part of *Ammotrypane aulogaster*, H. R., a peculiar modification of the body-wall exists, in the form of a constriction between the dorsal and ventral longitudinal muscles. The intermediate pedicle is formed apparently by the metamorphosed vertical muscles, which have coalesced over the nerve cords. The oblique muscles pass from the outer edge of the ventral longitudinal to the middle line below the nerve-area. A small and indistinct neural canal appears toward the upper part of the latter. In *Ophelia limacina*, H. R., the cords in the smoothly rounded anterior third lie in the middle of the long interval between the ventral longitudinal muscles. The oblique are inserted far outside the cords. Externally are a granular area, a series of transverse fibres, and the cuticle. Toward the posterior part of the body, where the ventral ridges are well-developed, the cords, by an intricate change in the relationships of the muscles, get above the median (ventral) insertion of the vertical, the fibres of which extend outward and somewhat downward into each pedicle. The true oblique is with difficulty seen; but it appears to be a

slender muscle passing from the outer border of the ventral longitudinal. In many sections a neural canal is seen somewhat above the middle of the area. A muscle passes from the bristle-tuft upward in a slanting direction through each pedicle to the raphe below the nerve. In *Travisia Forbesii*, Johnst., the nerve-trunks are situated between, and somewhat above, the insertions of the oblique muscle. Externally are the circular coat, a translucent basement-tissue, a granular hypoderm, and a very thin cuticle.

Scalibregmidæ.—In *Eumenia Jeffreysii* the cords in the anterior region occur toward the inner aspect of the thick layer of translucent basement-tissue, below and between the insertions of the two long oblique muscles. Toward the middle of the body the trunks still indent the basement-tissue, the circular muscular coat forming their inner boundary. A series of strong transverse fibres occurs at intervals as an arch over the cords. In *Scalibregma inflatum*, H. R., the cords (in the posterior region) lie below the circular muscular fibres, which occur beneath the commissure of the oblique in the middle line. Externally are the hypoderm and cuticle.

Telethusæ.—In the anterior third of *Arenicola marina*, L., the cords form a comparatively small ovoid mass—clasped by the great longitudinal muscles, and connected with the circular coat by a narrow granular pedicle.

Sphærodoridæ.—While there is a very thick cuticle in *Ephesia gracilis*, H. R., the hypoderm is slightly developed below the nerves. The oblique muscles pass from the inferior border of the bristle-tufts to the outer margin of the nerve-trunks.

Chloræmidæ.—In *Trophonia plumosa*, O. F. Müller, the nerves lie above the median decussation of the oblique muscles, the cords being distinct in the intervals between the ganglia. Externally are also the circular coat, a narrow hypoderm and a roughly papillose cuticle.*

Chætopteridæ.—In *Chætopterus norvegicus*, Sars., the cords are placed wide apart in front, in consonance with the peculiarly modified muscular arrangements of the body-wall. They are hypodermic. By the great increase of the median system of muscles

* Next the Chloræmidæ Dr Malmgren places the Sternaspididæ, but in structure Sternaspis is Gephyrean.

the homologues of the ventral longitudinal are pushed outward. The oblique preserve their usual relations with the latter and the nerve-cords, over which their ventral termination occurs. Another muscle, probably the homologue of the transverse, passes from the termination of the oblique inward to the middle line on each side. As the late M. Claparède has shown, the cords approach each other in the posterior region in *Chætopterus*, while in *Telepsavus* they remain separate throughout.*

SPIONIDÆ.—As pointed out last session,† the cords in this family are hypodermic in position. Their relation to the ventral insertions of the oblique muscles are also well shown, since they follow the latter in their gradual progress inward from the sides of the body in front until the cords touch and the muscles meet over them. The neural canals are largely developed.

CIRRATULIDÆ.—In *Cirratulus cirratus*, O. F. Müller, the cords in the anterior region lie in the median line ventrally within a thick hypodermic area. The oblique muscles are inserted at the summit of the area, and the sides of the ventral longitudinal muscles overlap its upper arch. The circular muscular coat is continuous over (internal to) the cords. In *Dodecaceria concharum*, Œrst., a thick median mass of blackish hypoderm protects the cords, and the relations of the oblique and longitudinal ventral muscles are similar.

HALELMINTHIDÆ.—This family approaches the Lumbricidæ in having the nerve-cords placed within the great and nearly continuous longitudinal muscles of the body-wall. Externally (in the ventral region of *Capitella capitata*, Fabr.) are in addition a dense circular muscular coat, a thin layer of longitudinal fibres, basement-tissue, hypoderm, and cuticle. Anteriorly there are two divisions of the ventral longitudinal muscular fibres under the nerves, but in the middle region of the body they have coalesced into a single mass. A neural canal occurs superiorly in the ovoid nervous area. Two vertical muscles bound those above which the nerves lie, but the oblique are not recognisable.

MALDAIDNÆ.—In the anterior region of *Praxilla prætermissa*, Mgrn., the nerve-cords lie beneath (outside) the circular muscular

* Annélides Sédent. p. 127, &c.

† Proceed. R.S.E., vol. ix. No. 94, p. 124, &c.

coat, at the gap between the ventral longitudinal muscles. The area is small and flattened, and has a large neural canal in the centre superiorly. The oblique muscles are attached above the area. A similar arrangement occurs in *Nicomache lumbricalis*, Fabr., and in a large Canadian species of the same genus.

Ammocharidæ.—About a quarter of an inch behind the snout the nerve-area in *Owenia filiformis*, D. Ch., forms an ovoid mass outside the tough basement-tissue bounding the great longitudinal muscular layer. The area is entirely hypodermic, and as very little of this tissue remains in the majority of the preparations, the cords are often bare. The oblique muscles are not visible, and the longitudinal are only separated in the median line dorsally and ventrally.

Hermellidæ.—In *Sabellaria spinulosa*, R. Leuckart, the cords remain quite separate throughout their entire length. Anteriorly each is placed in the substance of the great ventral muscle, near its upper and inner border. No distinct oblique muscles appear in this form. Posteriorly the nerves occupy the same relative position in the diminished muscles, and each has a large neural canal at its inner border.

Amphictenidæ.—In *Cystenides hyperborea*, Mgrn., the united cords in the anterior region occur as an ovoid mass over the transverse muscular fibres in the ventral median line. The large oblique muscles are widely separated from the nerve-trunks.

Ampharetidæ.—In *Amphicteis Gunneri*, from Canada, the nerves (in transverse section of the body-wall) appear as two minute separate bodies enveloped in a common neurilemma lying in the thick hypoderm of the median ventral region. Internally are the fibres of the circular coat and the insertions of the oblique and vertical muscles.

Terebellidæ.—Anteriorly in *Terebella nebulosa*, Mont., the nerve-cords are placed outside the transverse band of fibres (part of the circular coat) between the oblique muscles, and therefore are hypodermic. The same arrangement occurs posteriorly. In *Polycirrus aurantiacus*, Gr., the cords have the same relative position, only they are considerably larger—a feature of interest in connection with the phosphorescent properties of the species. Within the circular coat is a small median longitudinal muscle. In *Terebel-*

lides strœmi, Sars., the cords are small, and lie within the hypoderm and strong circular coat in a line between the ventral longitudinal muscles. The slender oblique are attached on each side of the trunks, and must be carefully distinguished from the much larger muscles which cut off the great lateral longitudinal muscle externally.

SABELLIDÆ.—In *Sabella pavonina*, Sav., the nerves occur in front as widely separated cords—each situated at the inner border of the ventral longitudinal muscle. A large neural canal lies above each. Externally are the circular muscular coat, the greatly thickened hypoderm and the cuticle. The feeble oblique muscles are partly inserted above the neural canal and partly into the basement-layer below and internal to the nerve. Proceeding backward, the neural canal increases very much in size, so that it occupies the whole area on each side of the central region, and presses the ovoid nerve-cord to the exterior. The canals appear to be filled with a slightly yellowish substance—consolidated in the preparations. The same relative position is maintained posteriorly, though the neural canal is smaller, and, as in front, placed superiorly. The arrangement in *Dasychone bombyx*, Dalyell, agrees in most respects with the foregoing.

Chone and *Euchone* seem to approach the next family in the arrangement of the nerve-cords and in the remarkable structure of the great longitudinal muscles. In *Chone infundibulum*, Mont., the nerve-trunks occur between the ventral longitudinal muscles, and placed much more closely together than in *Sabella*. A large neural canal lies over each nerve. Externally are the strong circular coat, the thick hypoderm, and the cuticle. The oblique muscles are attached to the sides of the median vessel above the neural canals, and to the summit of the latter on each side. While the cords are lost in the ganglia, the neural canals remain distinct. In *Euchone* a similar condition exists, for though the cords are distinct anteriorly they are closely approximated posteriorly.

ERIOGRAPHIDIDÆ.—In the anterior third of *Myxicola infundibulum*, Renier, the cords are approximated in the interval between the great longitudinal ventral muscles, and beneath the large blood-vessel. A neural canal appears to exist toward the superior border. The slender oblique muscles are inserted into the summit

of the nerve-area. Externally are circular coat, hypoderm, and cuticle. Posteriorly the cords are connate, and a single neural canal of considerable size exists superiorly.

Serpulidæ.—In *Protula protensa* the cords anteriorly are situated under the great longitudinal dorsal muscles as widely separated trunks. They then fall into position and approach the increasing longitudinal ventral muscles—a large neural canal being at the inner border. Posteriorly the neural canal is proportionally larger than in front, but the relations of the nerves are similar. At their commencement in front the nerve-cords are internal, and have beneath them all the tissues under the dorsal muscles. After reaching the inner border of the ventral longitudinal muscles they have externally (*i.e.*, inferiorly) a thin layer of longitudinal muscular fibres, the circular coat, hypoderm, and cuticle. A transverse band bounds them superiorly. Posteriorly the first-mentioned (longitudinal) layer is absent externally. In *Serpula vermicularis*, L., a similar arrangement occurs, but the neural canal is external, that is, next the ventral longitudinal muscle, and the thin longitudinal stratum is internal.

4. On the Application of Graphic Methods to the Determination of the Efficiency of Machinery. By Professor Fleeming Jenkin.

(*Abstract.*)

The general scope of the paper was to show how by graphic methods we might find the relation between effort exerted at one part of a machine and the resistance overcome at another part. It was shown that for a given machine at any instant a linear frame might be substituted, such that the stresses in the links corresponded to the pressures at the joints between the elements of the machine, including the pressures due to driving effort and resistance. Numerous examples were given of the application of this frame, called the dynamic frame, to the solution of the above problem, the friction, inertia, and weight of all the parts being taken rigidly into account.

5. On Professor Tait's Problem of Arrangement. By Thomas Muir, M.A.

The problem in question is—To find the number of possible arrangements of a set of n things, subject to the conditions that the first is not to be in the last or first place, the second not in the first or second place, the third not in the second or third place, and so on.

A little consideration serves to show that we may with advantage shift the ground of the problem to the theory of determinants. For the sake of definiteness take the case of *five* things, A, B, C, D, E. Here A may be in the second, third, or fourth places only; B in the third, fourth, or fifth places only; and similarly of the others—a result which we may tabulate thus:—

$$\begin{array}{ccccc} . & A & A & A & . \\ . & . & B & B & B \\ C & . & . & C & C \\ D & D & . & . & D \\ E & E & E & . & . \end{array}$$

an A being written in the places which it is possible for A to occupy, and a dot signifying that the letter found in the same line with it may not occupy its place. Hence, to obtain the various arrangements, we see that for the first place we may have any letter that is in the first column; for the second place any letter that is in the second column, provided it be not in the same line with the letter taken from the first column; for the third place, any letter that is in the third column, provided it be not in the same line with either of the letters previously taken, and so on. This law of formation, however, is identical with that in accordance with which the terms of a determinant are got from the elements of the matrix; so that the problem we are concerned with is transformed into this: Find the number of terms of the determinant of the nth order of the form,—

$$\left|\begin{array}{ccccc} \cdot & * & * & * & \cdot \\ \cdot & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{array}\right|$$

where the dots and asterisks denote zero elements and non-zero elements respectively.

In the solution of this, determinants of a set of other forms require to be considered, viz.,—

$$\left|\begin{array}{ccccc} \cdot & * & * & * & * \\ \cdot & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{array}\right| , \left|\begin{array}{ccccc} \cdot & * & * & * & * \\ * & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{array}\right| , \left|\begin{array}{ccccc} \cdot & * & * & * & * \\ \cdot & \cdot & * & * & * \\ * & * & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{array}\right| , \text{\&c.}$$

in all of which the elements of the main diagonal are zero; the elements of the adjacent minor diagonal being in the first also all zero, in the second all zero except the first element, in the third all zero except the second element, and so on. Let us denote the number of terms in these when they are of the nth order by

$$\chi_0(n)\,,\ \chi_1(n)\,,\ \chi_2(n)\,,\ \ldots .$$

and let the number of terms sought be

$$\Psi(n)\,.$$

Further, let any determinant-form with dots and asterisks stand for *the number of terms* in such a determinant; the four forms above, for example, being thus symbols equivalent to $\Psi(5)$, $\chi_0(5)$, $\chi_1(5)$, $\chi_2(5)$, respectively.

Beginning with the first of the χ forms we see it to be transformable into

$$\begin{vmatrix} \cdot & * & * & * & \cdot \\ \cdot & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & * & * \\ * & \cdot & \cdot & * \\ * & * & \cdot & \cdot \\ * & * & * & \cdot \end{vmatrix}$$

and the second term of this may in like manner be changed into

$$\begin{vmatrix} \cdot & \cdot & * & * \\ * & \cdot & \cdot & * \\ * & * & \cdot & \cdot \\ \cdot & * & * & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & * & * \\ \cdot & \cdot & * \\ * & \cdot & \cdot \end{vmatrix}$$

and the second term of this again into

$$\begin{vmatrix} \cdot & * & \cdot \\ \cdot & \cdot & * \\ * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot \\ * & \cdot \end{vmatrix}$$

the second term of which is zero. Hence we have the result,

$$\chi_0(n) = \Psi(n) + \Psi(n-1) + \Psi(n-2) + \ldots + \Psi(3), \quad . \quad . \quad . \quad (a)$$

Of the other χ forms it is clear to begin with that

$$\chi_1 = \chi_{n-1}, \qquad \chi_2 = \chi_{n-2}, \quad \ldots .$$

and treating the distinct cases as we have treated χ_0, we equally readily see that

$$\begin{aligned} \chi_1(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) \\ \chi_2(n) &= \chi_0(n) + \chi_0(n-1) + \chi_1(n-2) \\ \chi_3(n) &= \chi_0(n) + \chi_0(n-1) + \chi_2(n-2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \chi_{n-2}(n) &= \chi_0(n) + \chi_0(n-1) + \chi_1(n-2) \\ \chi_{n-1}(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) \end{aligned}$$

which, on eliminating $\chi_1, \chi_2, \ldots$ from the right-hand members, become

$$\chi_1(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2)$$

$$\chi_2(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4)$$

$$\chi_3(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) + \chi_0(n-6)$$

....

$$\chi_{n-2}(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4)$$

$$\chi_{n-1}(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2)\,.$$

Here the second of the series of right-hand members has two terms more than the first, the third two terms more than the second, and so on until we approach the middle of the series, when, if n be odd, the two middle right-hand members are found to be the same as the one preceding or the one following them, the whole four ending thus—

$$\ldots\ldots + \chi_0(5) + \chi_0(4) + \chi_0(3)\,;$$

and if n be even, the one middle right-hand member is found to be greater by unity than the one preceding or the one following it, and to end thus—

$$\ldots\ldots + \chi_0(5) + \chi_0(4) + \chi_0(3) + 1\,,$$

the 1 arising from the fact that the above process of reduction, in the case of $\chi_{\frac{1}{2}n}(n)$, leads us finally, not to

$$\left|\begin{array}{ccc} \cdot & * & * \\ \cdot & \cdot & * \\ * & \cdot & \cdot \end{array}\right| + \left|\begin{array}{cc} * & * \\ \cdot & \cdot \end{array}\right|, \text{ but to } \left|\begin{array}{ccc} \cdot & * & * \\ \cdot & \cdot & * \\ * & \cdot & \cdot \end{array}\right| + \left|\begin{array}{cc} \cdot & * \\ * & \cdot \end{array}\right|$$

i.e., to $\chi_0(3) + 1$.

Returning now to the Ψ form, and taking

$$\left|\begin{array}{ccccc} \cdot & * & * & * & \cdot \\ \cdot & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{array}\right|$$

we transform it into

$$\begin{vmatrix} \cdot & * & * & * \\ * & \cdot & * & * \\ * & \cdot & \cdot & * \\ * & * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & * & * \\ * & \cdot & * & * \\ * & * & \cdot & * \\ * & * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & * & * \\ * & \cdot & \cdot & * \\ * & * & \cdot & * \\ * & * & * & \cdot \end{vmatrix}$$

the middle term of which becomes by transposition of the first two rows, and the subsequent transposition of the first two columns,

$$\begin{vmatrix} \cdot & * & * & * \\ \cdot & \cdot & * & * \\ * & * & \cdot & * \\ * & * & \cdot & \cdot \end{vmatrix}$$

Consequently we have

$$\Psi(5) = \chi_1(4) + \chi_2(4) + \chi_3(4)\,,$$

and it is easily seen that a similar transformation is possible in every case, giving

$$\Psi(n) = \chi_1(n-1) + \chi_2(n-1) + \chi_3(n-1) + \ldots . + \chi_{n-2}(n-1)\,.$$

Expressing χ_2, χ_3, ... in terms of χ_0 by means of what precedes, we have

$$\begin{aligned} \Psi(n) = \;& \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) \\ & + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) \\ & + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3)\,, \end{aligned}$$

and now using (a), to express χ_0 in terms of Ψ, we find

$$\begin{aligned}\Psi(n) = \;& \Psi(n-1)+2\Psi(n-2)+3\{\Psi(n-3)+\ldots\ldots\ldots\ldots\ldots+\Psi(3)\}\\ & +\Psi(n-1)+2\Psi(n-2)+3\Psi(n-3)+4\Psi(n-4)\\ & \qquad\qquad +5\{\Psi(n-5)+\ldots\ldots\ldots\ldots\ldots+\Psi(3)\}\\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ & +\Psi(n-1)+2\Psi(n-2)+3\Psi(n-3)+4\Psi(n-4)\\ & \qquad\qquad +5\{\Psi(n-5)+\ldots\ldots\ldots\ldots\ldots+\Psi(3)\}\\ & +\Psi(n-1)+2\Psi(n-2)+3\{\Psi(n-3)+\ldots\ldots\ldots\ldots\ldots+\Psi(3)\}\end{aligned}$$

where, on the first line the coefficient of the third and all the following terms is 3, on the second line the coefficient of the fifth and all the following terms is 5, on the third line the coefficient of the seventh and all the following terms is 7, and so on, the middle term (when such occurs) having a 1 superadded.

Hence, for the determination of $\Psi(n)$ when $\Psi(n-1)$, $\Psi(n-2)$, ... are known, we have

$$\begin{aligned}\Psi(n) = (n-2)\Psi(n-1)+(2n-\;\;4)\Psi(n-2)&+(3n-\;\;6)\Psi(n-3)\\ +(4n-10)\Psi(n-4)&+(5n-14)\Psi(n-6)\\ +(6n-20)\Psi(n-6)&+(7n-26)\Psi(n-7)\\ \ldots\ldots\ldots\ldots\ldots\ldots&\ldots\ldots\ldots\ldots\ldots\ldots\\ +\frac{1-(-1)^n}{2}&,\end{aligned}$$

where the coefficients proceed for two terms with the common difference $n-2$, for the next two terms with the common difference $n-4$, for the next two terms with the common difference $n-6$, and so on.

And as it is self-evident that $\Psi(2)=0$, we obtain

$$\begin{aligned}\Psi(3) &= 1\Psi(2)+1 &&= 1\\ \Psi(4) &= 2\Psi(3) &&= 2\\ \Psi(5) &= 3\Psi(4)+6\Psi(3)+1 &&= 13\\ \Psi(6) &= 4\Psi(5)+8\Psi(4)+12\Psi(3) &&= 80\\ \Psi(7) &= 5\Psi(6)+10\Psi(5)+15\Psi(4)+18\Psi(3)+1 &&= 579\\ \Psi(8) &= 6\Psi(7)+12\Psi(6)+18\Psi(5)+22\Psi(4)+26\Psi(3) &&= 4738\end{aligned}$$

and so forth.

To the foregoing Professor Cayley has kindly made the following additions :—

The investigation may be carried further: writing for shortness u_3, u_4, &c., in place of $\Psi(3)$, $\Psi(4)$, &c., the equations are

$$\begin{aligned} u_3 &= 1\,, \\ u_4 &= 2u_3\,, \\ u_5 &= 3u_4 + 6u_3 + 1\,, \\ u_6 &= 4u_5 + 8u_4 + 12u_3 \\ u_7 &= 5u_6 + 10u_5 + 15u_4 + 18u_3 + 1\,, \end{aligned}$$

and hence assuming

$$u = u_3 + u_4 x + u_5 x^2 + u_5 x^3 + u_7 x^4 \ldots$$

we have

$$\begin{aligned} u = \frac{1}{1-x^2} &+ u_3(2x + 6x^2 + 12x^3 + 18x^4 + \ldots \\ &+ u_4(3x^2 + 8x^3 + 15x^4 + 22x^5 + \ldots \\ &+ u_5(4x^3 + 10x^4 + 18x^5 + 26x^6 + \ldots \\ &+ u_6(5x^4 + 12x^5 + 21x^6 + 30x_7 + \;; \end{aligned}$$

and hence forming the equation

$$\begin{aligned} u'\frac{x^2}{(1-x)^2} = \; & u_4(x^2 + 2x^3 + 3x^4 + 4x^5 + \ldots) \\ & + u_5(2x^3 + 4x^4 + 6x^5 + 8x^6 + \ldots) \\ & + u_6(3x^4 + 6x^4 + 9x^6 + 12x^7 + \ldots)\,; \end{aligned}$$

where u' denotes $\frac{du}{dx}$, we have

$$\begin{aligned} u - u'\frac{x^3}{(1-x)^2} &= \frac{1}{1-x^2} + (u_3 + u_4 x + u_5 x^2 \ldots)(2x + 6x^2 + 12x^3 + 18x^4 \ldots) \\ &= \frac{1}{1-x^2} + u(2x + 6x^2 + 12x^3 + 18x^4 + \ldots)\,; \end{aligned}$$

or, what is the same thing,

$$u - u'\frac{x^2}{(1-x)^2} = \frac{1}{1-x^2} + u\left\{\frac{2x}{(1-x)^3} - \frac{2x^4}{(1-x)^3(1+x)}\right\}\,;$$

that is,

$$\left\{1-\frac{2x}{(1-x)^3}+\frac{2x^4}{(1-x)^3(1+x)}\right\}u-\frac{x^2}{(1-x)^2}u'=\frac{1}{1-x^2}.$$

This equation may be simplified : write

$$u=-\frac{1-x^2}{x^4}Q,\ =\left(-\frac{1}{x^4}+\frac{1}{x^2}\right)Q,$$

then

$$u'=\left(\frac{4}{x^5}-\frac{2}{x^3}\right)Q+\frac{1-x^2}{x^4}Q',$$

and the equation is

$$\left\{-\frac{1-x^2}{x^4}+\frac{2}{x^3}\frac{1+x}{(1-x)^2}-\frac{2}{(1-x)^2}\right.$$
$$\left.-\frac{4}{x^3}\frac{1}{(1-x)^2}+\frac{2}{x(1-x)^2}\right\}Q+\frac{1+x}{(1-x)x^2}Q'=\frac{1}{1-x^2};$$

that is,

$$\left\{-\frac{1}{x^4}+\frac{1}{x^2}-\frac{2}{x^3(1-x)^2}+\frac{2}{x^2(1-x)^2}+\frac{2}{x(1-x)^2}-\frac{2}{(1-x)^2}\right\}Q$$
$$+\frac{1+x}{(1-x)x^2}Q'=\frac{1}{1-x^2},$$

viz., this is

$$\left\{-\frac{(1-x)^2}{x^4}+\frac{(1-x)^2}{x^2}-\frac{2}{x^3}+\frac{2}{x^2}+\frac{2}{x}-2\right\}Q+\frac{1-x^2}{x^2}Q'=\frac{1-x}{1+x},$$

that is

$$\left\{-\frac{1}{x^4}+\frac{2}{x^2}-1\right\}Q+\frac{1-x^2}{x^2}Q'=\frac{1-x}{1+x};$$

or

$$-\frac{(1-x^2)^2}{x^4}Q+\frac{1-x^2}{x^2}Q'=\frac{1-x}{1+x};$$

or finally,

$$Q\left(1-\frac{1}{x^2}\right)+Q'=\frac{x^2}{(1+x)^2},$$

giving

$$Q=e^{-\left(x+\frac{1}{x}\right)}\int\frac{x^2}{(x+1)^2}e^{x+\frac{1}{x}}dx,$$

and thence

$$u = \frac{x^2-1}{a^4} e^{-\left(x+\frac{1}{x}\right)} \int \frac{x^2}{(x+1)^2} e^{\left(x+\frac{1}{x}\right)} dx \, ,$$

which is the value of the generating function

$$u = u_3 + u_4 x + u_5 x^2 + \&c.$$

But for the purpose of calculation it is best to integrate by a series the differential equation for Q: assume

$$Q = -q_3 x^4 - q_4 x^5 - q_5 x^6 - \ldots$$

then we find

$$\begin{aligned} q_4 &= 4q_3 \qquad\quad - 2, \\ q_5 &= 5q_4 + q_3 + 3, \\ q_6 &= 6q_5 + q_4 - 4, \\ q_7 &= 7q_6 + q_5 + 5, \\ &\vdots \\ q_n &= nq_{n-1} + 9_{n-2} + (-)^{n-1}(n-2) \, . \end{aligned}$$

We have thus for $q_3, q_4, q_5 \ldots$ the values 1, 2, 14, 82, 593, 4820, and thence

$$u = (1-x^2)(1 + 2x + 14x^2 + 82x^3 + 593x^4 + 4820x^5 + \ldots) \, ,$$

viz., writing

$$\begin{array}{rrrrrr} 1 & 2\,. & 14 & 82 & 593 & 4820\ldots \\ & & -1 & -2 & -14 & -82 \\ \hline \end{array}$$

the values of $u_3, u_4 \ldots$ are $1 \, , \, 2 \, , \, 13 \, , \, 80 \, , \, 579 \, , \, 4738 \ldots$ agreeing with the results found above.

In the more simple problem, where the arrangements of the n things are such that no one of them occupies its original place, if u_n be the number of arrangements, we have

$$\begin{aligned} u_2 &= 1 & &= 1 \\ u_3 &= 2u_2 & , &= 2 \\ u_4 &= 3(u_3 + u_2) & , &= 9 \\ u_5 &= 4(u_4 + u_3) & , &= 44 \\ &\vdots \\ u_{n+1} &= n(u_n + u_{n-1}) \, , \end{aligned}$$

and writing

$$u = u_2 + u_3 x + u_4 x^2 + \dots$$

we find

$$u = 1 + (2x + 3x^2)u + (x^2 + x^3)u' ;$$

that is

$$(-1 + 2x + 3x^2)u + (x^2 + x^3)u' = -1 ;$$

or, what is the same thing,

$$u' + \left(\frac{3}{x} - \frac{1}{x^2}\right)u = -\frac{1}{x^2(1+x)},$$

whence

$$u = x^{-3} e^{-\frac{1}{x}} \int \frac{-x}{1+x} e^{\frac{1}{x}} dx ,$$

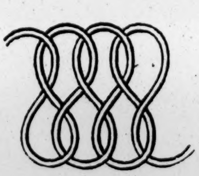
but the calculation is most easily performed by means of the foregoing equation of differences, itself obtained from the differential equation written in the foregoing form,

$$(-1 + 2x + 3x^2)u + (x^2 + x^3)u' = -1 .$$

6. On Amphicheiral Forms and their Relations. By Professor Tait.

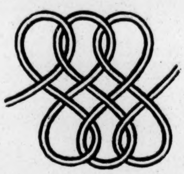
(*Abstract.*)

If a cord be knotted, any number of times, according to the pattern below

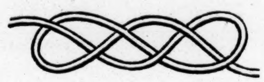


it is obviously *perverted* by simple *inversion*. Hence, when the free ends are joined it is an amphicheiral knot. Its simplest form is that of 4-fold knottiness. All its forms have knottiness expressible as $4n$.

The following pattern gives amphicheiral knots of knottiness $2+6n$.



And on the following pattern may be formed amphicheiral knots of all the orders included in $6n$ and $4+6n$.

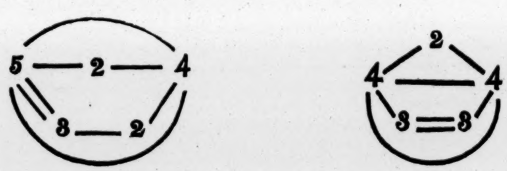


Among them these forms contain all the even numbers, so that *there is at least one amphicheiral form of every even order.*

Many more complex forms are given in the paper, several of which are closely connected with knitting, &c.

In one of my former papers I gave examples of type-symbol which individually represent two perfectly different knots.

I now give examples of the same knot represented by type-symbols which have neither right nor left-handed parts in common. One of the most remarkable of these is



which can be analysed (but not separated) into a combination of the two forms of the 4-fold amphicheiral knot.

The following Gentlemen were elected Fellows of the Society:—

William Jolly, H.M. Inspector of Schools, Inverness.
Rob. Milner Morrison, 21 Manor Place.
John Gibson, Ph.D., 12 Greenhill Gardens.
Chas. E. Underhill, B.A., M.B., 8 Coates Crescent.
Charles Edward Wilson, M.A., LL.D., 19 Palmerston Place.
George Carr Robinson, East Preston Street.

Monday, 16th April 1877.

PROFESSOR KELLAND, Vice-President, in the Chair.

The following Communications were read:—

1. On the Toothing of Un-round Discs which are intended to roll upon each other. By Edward Sang.

(*Abstract.*)

This paper contained an extension to discs of any shape whatever of the principles explained in the "New General Theory of the Teeth of Wheels," as applicable to circular discs.

2. On the Mineralogy of Scotland.—Chapter II. By Professor Heddle.

In this chapter Professor Heddle submitted the results of the analyses of Orthoclase from fifteen localities; of Albite from four; of Oligoclase from eight; of Labradorite from eleven; of Andesiel from five; of Anorthite from three; and of Latrobite from two.

The three last minerals being now, for the first time, recognised as British species.

Dr Heddle also described a peculiar association of Orthoclase with Oligoclase, in crystals nearly of the form of the former, from certain localities; he drew the conclusion from his researches that the above felspars are all well individualised, if not by physical, at least by chemical characters; while they are probably more or less special to certain rocks.

3. Least Roots of Equations. By J. D. Hamilton Dickson.

Monday, 7th May 1877.

DAVID STEVENSON, Esq., Vice-President, in the Chair.

The following Communications were read:—

1. On new and little-known Fossil Fishes from the Edinburgh District. No. III. By Dr R. H. Traquair.

2. On Ocean Circulation. By John Aitken.

It is with extreme reluctance that I venture to disturb the present repose of the much-contested field of ocean circulation. My object is not, however, to provoke discussion on the general theory of ocean circulation, as I am sure all will agree in thinking that the subject has already been discussed far beyond the point at which it is likely to be benefited by discussion. My object is simply to call attention to certain influences at work in the ocean, the effects of which seem to have been totally overlooked.

The first of them to which I wish to refer is the influence of the winds on the ocean. The extreme holders of the wind theory of ocean circulation consider that the action of the wind is quite sufficient to account for all the currents which we find in the ocean. That the wind is *a* cause of ocean currents no one can doubt. If we examine a lake when the wind is blowing over it, we shall find that the plants growing in the shallow water near the surface are all bending in the direction of the wind, indicating that there is a current at the surface flowing in the direction of the wind,—the appearance of the bending plants in the lake reminding one of a slow-running river. To supply the water for this surface current there must, of course, be another current, flowing in the opposite direction underneath. The lake and the ocean are not, however, parallel cases. In the case of the lake, the wind is blowing in the same direction all over it, so that the return current is forced to flow underneath the surface, as it cannot get back any other way; whereas, in the ocean, the wind blows in one direction at one part,

and in a different direction at another part, and the question now comes to be: What is the effect of the difference in the two cases? If the wind does not blow in the same direction at all parts over the surface of the water, will the return current flow underneath the surface current, or will it return by some other route?

It is extremely difficult to get a satisfactory experimental answer to this question. The following attempts were, however, made:—A trough with glass sides was filled with water, the water being well stirred just before the experiment was made, to prevent difference of density due to temperature having any effect on the result. When all was again at rest, a solution of colouring matter was dropped into the water at different points. Each drop of colouring matter, as it sunk to the bottom, left a vertical coloured streak in the water. A jet of air urged by a pair of bellows was now directed along the surface of the water, so as to act on only a small part of the surface, near the middle of the breadth of the trough. The upper part of the coloured streaks, situated underneath the air-jet, at once indicated a current in the surface water in the direction of the air-jet. The return current did not, however, flow *underneath* the air-driven current, as in the case of the lake, but the air-driven current divided in two at the far end of the trough, and *flowed back on the surface,* one current on each side of the air-driven current.*

When the air current was first started, the water currents were confined to the surface; but after the motion had been kept up some time, the depth of the currents gradually increased, till all the water in the trough was in motion,—the direction of the motion of the water at any part of the bottom being the same as at the surface vertically over it. The water showed not the smallest tendency to take up a "vertical" circulation, similar to the circulation produced by difference of density. The surface water simply circulated to different parts of the surface, and the bottom water to different parts of the bottom, almost the whole of the motion taking

* The trough in which the experiment is made must have sloping and not vertical ends, because if the wind-driven current strikes against a vertical surface, it raises a "head," which causes a vertical current to descend at the end of the trough, in addition to the two return surface currents.

place in a horizontal direction. In order to get quit, as much as possible, of the effects of friction on the sides and bottom of the vessel, the experiment was repeated in a pond, with a like result.

From these experiments we may conclude that, save under very exceptional circumstances, the wind can only give rise to a horizontal circulation of the oceanic waters,—the exceptional cases being, when the wind-driven current is deflected by the irregularities in the outline of the bed of the ocean, or strikes against a deep and nearly vertical coast; or when the north wind drives the waters of the Antarctic Sea against the great barrier of ice-cliffs which surround the south pole. The depth to which the vertical currents will descend in these cases will depend much on the relative densities of the water in the currents and of the water surrounding them.

It may possibly be objected that we are not entitled to come to any conclusion, from experiments made on so small a scale, as to what takes place in the ocean. Such objections would be perfectly valid, if there was not some evidence, in the conditions we find in the ocean, to support these conclusions. If the return currents in the ocean flowed underneath the wind-driven surface currents, then we should be perfectly justified in expecting some evidence of their presence. For instance, we would expect that the water near the bottom, underneath the wind-driven currents near the equator, would be hotter than the water at corresponding depths at other parts of the ocean. An examination, however, of the temperature sections of the Atlantic Ocean, taken by the "Challenger," shows no evidence whatever of the return current by this route; we are therefore compelled to conclude that the water must return by the surface, and that the *wind* does only produce *horizontal currents*, and therefore cannot account for the presence of the cold water which we find all over the bottom of the ocean, from the poles to the equator.

The second point I wish to refer to, is the effect of these wind-driven surface currents on the cold water underneath them. I have said that almost the whole of the motion of the wind-driven currents takes place in a horizontal direction. Such was the general result given by the experiment. But, in addition to this, there is another point to which I must refer. The wind driven

horizontal currents have an influence on the water underneath them which, for the sake of clearness, I have reserved for separate consideration here. Let us draw an imaginary section across a wind-driven current, at a point near its source, that is, near where the wind begins to act on it. And let us imagine another section of this same current, at a point some distance farther "down" the stream. As the current is acted upon by the wind between these two sections, it will be much deeper at the second section than at the first, and will also be going at a greater velocity. There will, therefore, be much more water passing the second section than passed the first, and the water necessary to supply this growing stream must be supplied to it between the two imaginary sections. The result is, part of the necessary supply rushes in at the sides, but part of it rises from the still water underneath the surface stream. This lifting of the deeper water by the surface current was very evident in the experiment already referred to, so long as the surface current was shallow, and gradually became less, as might be expected, when the current deepened.

From these considerations, we are naturally led to expect that the hot surface water at those parts of the ocean over which winds are constantly blowing, will be much reduced in depth, and that the cold bottom water will be found at a less depth underneath these surface currents than at any other part of the ocean. Part of this cold water will, in all probability, get mixed up with the bottom water of the surface current. And further, we would expect that this wind-driven hot surface water, after it passes beyond the windy regions, will gradually lose its motion and increase in depth.

These expectations are in a remarkable manner supported by the evidence of the temperature sections of the Atlantic taken by the "Challenger," and given in Dr Carpenter's paper in the "Proceedings of the Royal Geographical Society," vol. xviii. No. iv., 1874. Take, for instance, the section between Tenerife and St Thomas. In the first part of the journey there is not much alteration in the relative position of the isotherms, but after crossing the Tropic of Cancer and getting into the region of the north-east trade winds, the hot surface isotherms gradually approach each other, and the isotherm of 40°, which off the coast of Tenerife was

at a depth of nearly 1000 fathoms, rises to about 700 fathoms before it arrives at St Thomas. Again, in the passage north from St Thomas to Bermuda, and on to Halifax and New York, the temperature sections show that after getting out of the region of the trade winds, the drifted hot surface water has gradually lost its motion and increased in depth. This refers to the great mass of ocean water, and not to the comparatively shallow Gulf-Stream. For instance, the isotherm of 60°, which at St Thomas was found at a depth of only 200 fathoms, was found at a depth of 330 fathoms for hundreds of miles all round Bermuda, notwithstanding a considerable reduction in the temperature of the surface water.

The temperature sections of the South Atlantic do not illustrate these points so well as the temperature sections of the North Atlantic, partly because a very large part of the hot surface water of the south equatorial current does not return to the South Atlantic, but is driven into the North Atlantic, and partly on account of the great amount of cold surface water of the Antarctic drift, which gets mixed np with the return current; and further, the temperature sections are not taken at the best places for our present purpose. The only two available sections, however, point to the same conclusion as the North Atlantic sections. There is no section of the south equatorial current, but we may suppose it to be somewhat similar to the section taken between St Paul's Rocks and Pernambuco, which gives the isotherms of the branch of the south equatorial current which passes into the North Atlantic. If we compare this section with the part of the stream which has flowed southwards, as given in the section taken between Abrolhos Island and Tristan d'Acunha, we find that the hot surface water, in flowing southwards beyond the region of the south-east trade winds, has deepened, notwithstanding a considerable fall in the temperature of the surface water. The isotherm of 40° which was found at a depth of 300 fathoms off Pernambuco, sunk to a depth of between 400 and 500 fathoms between Abrolhos Island and Tristan d'Acunha. We might have expected that this hot surface water would have kept its depth all the way to the Cape of Good Hope. It, however, does not do so, probably on account of the cold surface water of the Antarctic drift.

I am aware Dr Carpenter has offered a different explanation of the

rising of the glacial water under the equator. He considers that the rising of the glacial water under the Line is due to the meeting of the Arctic and Antarctic underflows. That part of the effect is due to this cause is very probable, but when we consider the very great area of section of the under glacial currents, and the small amount of water that can be carried by them, it is evident that their rate of motion must be very slow, and it is very doubtful how far the whole phenomena can be explained in this way. Our doubts on this subject are somewhat confirmed by the consideration of the fact that the Arctic underflow is warmer than the Antarctic underflow, and would therefore—other things being equal—tend simply to *overflow* the Antarctic underflow, and not to rise vertically as Dr Carpenter supposes. That the wind-driven currents, so long as they are increasing in volume, have the power of drawing the bottom water upwards cannot be doubted, and I have already said was most marked in the experimental illustration. We should not, however, expect to find this lifting power so marked in the ocean, as the ocean currents are so much deeper and do not so rapidly increase in volume as in the experiment. And further, the bottom water in the ocean is denser than the surface water, and does not rise so easily as the bottom water in the experiment.

If we take the evidence of the temperature sections of the Atlantic on the subject, we shall find that they also point to the wind as one of the causes of the rising of the cold bottom water. If we take the section between Madeira and lat. 3° N. and long. 15° W, we find that the isotherm of 40°, which at Madeira lies at a depth of about 900 fathoms, rises to a depth of only 300 fathoms at the equatorial position, and further, neither this section nor the section between lat. 3° N. and long. 15° W. and Pernambuco, show the least evidence of the presence of Antarctic water so far north as the equator, in the eastern basin of the Atlantic. *This rising, then, of the glacial water on the eastern side of the Atlantic cannot therefore be due to the meeting of the glacial streams*, and in the absence of further evidence we may suppose it to be due to the wind-driven currents. We thus see, that though the great bulk of the wind-driven circulation is a horizontal one, yet there is also produced in a comparatively very small degree a modified vertical circulation.

It may here be asked, Is this lifting power of the surface currents sufficient to account for the vertical circulation which we find in the ocean? In all probability it is not. There seems to be no reason why this vertically rising current under the equator should draw its supplies from the furthest limits of the ocean, which it would require to do to explain the conditions we find existing. Yet there can be no doubt but that these horizontal surface currents really do assist in producing a vertical circulation.

3. On a New Investigation of the Series for the Sine and Cosine of an Arc. By Edward Sang.

The sines of the successive equidifferent arcs form a progression having for its general character the relation

$$\phi_{n-1} - 2\phi n + \phi_{n+1} - \phi n \,.\, v\,,$$

and the properties of sines may be deduced from this general formula. Viewed in this light, the angular functions become cases only of more general ones.

If we suppose A, B, C to be three consecutive terms of such a progression we must have

$$A - 2B + C = vB\,,$$

from which, when three of the four quantities, A, B, C, v are given, the fourth may be found. Let then A and B, the first and second terms of the progression, and v the common coefficient, be given; the succeeding terms may be computed thus:—

$$\begin{array}{lll}
\phi 0 = & A & \\
 & -\ A & +B \\
 & & +Bv \\
\hline
\phi 1 = & & B \\
 & -A & +B(1+v) \\
 & -Av & +B(\ \ +2v+v^2) \\
\hline
\phi 2 = & -A & +B(2+v) \\
 & -A(1+v) & +B(1+3v+v^2) \\
 & -A(\ \ +2v+v^2) & +B(\ \ +3v+4v^2+v^3) \\
\hline
\end{array}$$

$$\phi 3 = -A(2+v) \qquad +B(3+4v+v^2)$$
$$-A(1+3v+v^2) \qquad +B(1+6v5v^2+v^3)$$

$$\phi 4 = -A(3+4v+v^2) \qquad +B(4+10v+6v^2+v^3),$$

from which it is obvious that the coefficient of $-A$ in the expression for ϕn, is a transcript of that of B in the preceding expression for $\phi(n-1)$. Hence, for the present, we may confine our attention to the latter.

The coefficients of B form the following progression:—

in $\phi 0$ 0
in $\phi 1$ 1
in $\phi 2$ $2+v$
in $\phi 3$ $3+4v+v^2$
in $\phi 4$ $4+10v+6v^2+v^3$
in $\phi 5$ $5+20v+21v^2+8v^3+v^4$
in $\phi 6$ $6+35v+56v^2+36v^3+10v^4+v^5$
in $\phi 7$ $7+56v+126v^2+120v^3+55v^4+12v^5+v^6$
in $\phi 8$ $8+84v+252v^2 \times 330v^3+220v^4+78v^5+14v^5+v^7$
in $\phi 9$ $9+120v+462v^2+792v^3+715v^4+364v^5+105v^6+16v^7+$ &c.

and in general

$$\text{in } \phi n \quad \frac{n}{1}+\frac{n-1}{1}\,\frac{n}{2}\,\frac{n+1}{3}\,v+\frac{n-2}{1}\,\frac{n-1}{2}\,\frac{n}{3}\,\frac{n+1}{4}\,\frac{n+2}{5}\,v^2 + \text{\&c.}$$

When v is positive the formulæ belong to the class of catenarian functions; when v is negative, to the circular ones.

If we put $\sin pa$ for $\phi 0$, $\sin \overline{p+1}\, a$ for $\phi 1$, and $-\text{chord } a$ for v, we obtain

$$\sin(p+n)a = -\sin pa\left\{\frac{n-1}{1}-\frac{n-2}{1}\,\frac{n-1}{2}\,\frac{n}{3}\text{ cho } a^2+\text{ \&c.}\right\}$$

$$+\sin(p+1)a\left\{\frac{n}{1}-\frac{n-1}{1}\,\frac{n}{2}\,\frac{n+1}{3}\text{ cho } a^2+\text{ \&c.}\right\}$$

and in thus putting $p=0$

$$\sin na = n\sin a\left\{1-\frac{n^2-1}{1.2}\,\frac{\text{cho } a^2}{3}+\frac{n^2-1}{1.2}\,\frac{n^2-4}{3.4}\,\frac{\text{cho } a^3}{5}-\text{\&c.}\right\}$$

And again writing $na = A$, $a = \frac{A}{n}$, this takes the form

$$\sin A = n \sin \frac{A}{n} \left\{ 1 - \frac{n^2 - 1}{1.2} \frac{1}{3} \left(\text{cho} \frac{A}{n} \right)^2 + \frac{n^2 - 1}{1.2} \frac{n^2 - 4}{3.4} \frac{1}{5} \left(\text{cho} \frac{A}{n} \right)^2 - \&c. \right\}$$

Now when n becomes indefinitely great, $n \sin \frac{A}{n}$ becomes A, so also $n \text{ cho } \frac{A}{n}$, wherefore $\sin A = \frac{A}{1} - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \&c.$

In order to obtain the series for the cosine we must put $pa = \frac{\pi}{2}$, and therefore $\sin (p + 1)a = \cos a$, which gives

$$\cos na = -1 \left\{ \frac{n-1}{1} - \frac{n-2}{1} \frac{n-1}{2} \frac{n}{3} \text{ cho } a^2 + \&c. \right\}$$

$$+ \cos a \left\{ \frac{n}{1} - \frac{n-1}{1} \frac{1}{2} \frac{n+1}{3} \text{ cho } a^2 + \&c. \right\}$$

and if, in this, we substitute for $\cos a$, its value $1 - \frac{1}{2} \text{ cho } a^2$,

$$\cos na = 1 - \frac{n}{1} \frac{n-2}{2} \text{ cho } a^2 + \frac{n-1}{1} \frac{n}{2} \frac{n+1}{3} \frac{n-4}{4} \text{ cho } a^4 \ \&c.$$

whence, proceeding as before,

$$\cos A = 1 - \frac{A^2}{1.2} + \frac{A^4}{1 \ldots . 4} - \frac{A^6}{1 \ldots .. 6} + \&c.$$

4. Note on the Bifilar Magnetometer. By J. A. Broun, F.R.S. Communicated by Professor Tait.

5. Addition to the paper "On the establishment of the Elementary Principles of Quaternions," by G. Plarr,—published in Vol. XXVII. of the Transactions of the Society. Communicated by Professor Tait.

6. Note on Mr Muir's Solution of a "Problem of Arrangement." By Professor Cayley.

This note has been printed along with Mr Muir's paper, *ante*, p. 382.

7. Preliminary Note on a New Method of Investigating the Properties of Knots. By Professor Tait.

As we cannot have knots in two dimensions, and as Prof. Klein has proved that they cannot exist in space of four dimensions, it would appear that the investigation of their properties belongs to that class of problems for which the methods of quaternions were specially devised. The equation

$$\rho = \phi(s),$$

where ϕ is a periodic function, of course represents any endless curve whatever. Now the only condition to which variations of this function (looked on as corresponding to *deformations* of the knot) is subject, is that *no two values of* ρ *shall ever be equal* even at a *stage* of the deformation. Subject to this proviso, ϕ may suffer any changes whatever—retaining of course its periodicity. Some of the simpler results of a study of this novel problem in the theory of equations were given,—among others the complete representation of any knot whatever by three closed plane curves, non-autotomic and (if required) non-intersecting.

The following Gentlemen were elected Ordinary Fellows of the Society:—

Robert A. Macfie, Dreghorn, Colinton.
William Stirling, Sc.D., M.D.

Monday, 21st May 1877.

Professor KELLAND, Vice-President, in the Chair.

The following Communications were read:—

1. On the Cranial Osteology of Rhizodopsis, and on some points in the Structure of Rhizodus. By Dr R. H. Traquair.

2. Notice of Recent Earthquake Shocks in Argyleshire in 1877. By David Stevenson, Civil Engineer.

Two earthquake shocks have lately occurred in Argyleshire of so decided a character that a description of their effects, as observed

at four of the Lighthouse stations on the west coast of Scotland, will, it is thought, be interesting to the Society.

The first shock occurred on the 11th March, and was observed at the Lighthouse station of Hynish in the island of Tyree, and at Sound of Mull, near Tobermory, the distance between the two places being about 34 statute miles.

The report from Tyree states:—"On the 11th current (March), at half-past 11 o'clock A.M., a smart shock of earthquake was felt all along the island; a great many people both heard the noise and at the same time felt the earth to tremble. It was heard and felt very distinctly at the station." Bar. 30·18 at 9 A.M.

That from the Sound of Mull says:—"On the 11th, at 11.30 A.M., this district was visited by a smart shock of earthquake. It began by a rumbling noise like distant thunder. When the noise was at its height the houses, and everything about them, shook, and the slates on the roof rattled. The shaking was not of long duration, but the noise was heard a considerable time before and after the trembling of the earth." Bar. 29·92 at 9 A.M.

The second shock, which seems to have been more severe, took place on the 23d April, and was observed at the island of Phladda, off Easdale, and at Lismore, at the eastern entrance to the Sound of Mull, the distance between the two stations being about 73 statute miles.

The report from Phladda states:—"At 3.40 A.M. the Principal Keeper on the watch felt a severe shock of earthquake. The tower and dwelling-houses shook very much. All the neighbouring islands felt it at the same time." Bar. 29·74 at 9 A.M.

At Lismore the lightkeeper describes the effect as follows (the lighthouse clock had been under repair):—"I beg leave to report that on the morning of the 23d, at 3.30 A.M., while I was standing on the grating inside the lightroom I felt a heavy shock on the tower, with a strange rumbling sound of noise which lasted some seconds, and made everything in the lightroom shake at an alarming rate. It awoke all the inmates of the dwelling-houses. Mr M'Leod jumped out of bed, thinking the tower had fallen, but afterwards thought it was a peal of thunder. I do not think it was thunder. I saw no lightning, and the wind was

light at the time. There is no damage done to anything about the station so far as I can see." Bar. 29·43 at 9 A.M.

These observations are valuable because of their trustworthiness as coming from wholly independent observers, and because few more sensible earthquake shocks have, so far as I know, been observed in Scotland. It is also remarkable that they do not appear to have been felt at any other lighthouse stations, although there are several others in the immediate vicinity. A record of them may therefore be useful to those engaged in seismic investigations.

3. Additional Remarks on Knots. By Professor Tait.

The author, in laying before the Society a revised and condensed version of the various papers recently communicated by him, took occasion to make some additional remarks. Of these only one need be given here. He pointed out that another fundamental term is requisite besides those already used viz., *Knots* and *Links*. For three endless cords may be inseparably entangled with one another, or locked together, even if no one of them be knotted and no two interlinked.

Monday, 4th June 1877.

DAVID STEVENSON, Esq., C.E., Vice-President, in the Chair.

The following Communications were read:—

1. On the Structure and Relations of the Genus Holopus. By Sir C. Wyville Thomson, F.R.S.

(*Abstract.*)

The "Challenger" Expedition had no opportunity of visiting Barbadoes, and this I regretted greatly, as Sir Rawson Rawson, who was at that time governor of the island, had paid great attention to the marine fauna, and was anxious to introduce us to his fine collection, which included many specimens of the rare and

remarkable forms for which the sea of the Antilles is famous. However, although the "Challenger" was not there in person, on her return Sir Rawson Rawson most kindly and liberally placed the finest of his specimens at my disposal for examination and description; and it is through his liberality that I now have it in my power to exhibit the very singular creature which is the subject of these notes. In 1837 M. Alcide d'Orbigny described and figured in the "Magasin de Zoologie," under the name of *Holopus*, a new recent genus of fixed Crinoids; and as the description of this distinguished palæontologist indicated an undescribed form of great interest, it was speedily reproduced in the "Annales des Sciences Naturelles" and in Wiegmann's "Archiv."

The specimen described by D'Orbigny, which was for a long time unique, was brought from Martinique by M. Sander-Rang. It subsequently passed into the possession of M. D'Orbigny, who described it under the name of *H. rangii*. D'Orbigny's account was very clear and intelligible, and his determination was fully borne out by his figures; and in Bronn's "Classen und Ordnungen des Thier-Reichs," published somewhere about 1861, the description and figures are repeated, and a distinct family, Holopidæ, is adopted for the reception of the single species. It is very singular that in the "Historie Naturelle des Zoophytes Echinodermes," by Dujardin and Hupé, published as a volume of the "Suites à Buffon" in 1862, the authors express their opinion that *Holopus* is not a Crinoid, but some totally different thing, probably a Cirriped, and they profess to have been unable to find D'Orbigny's specimen.

At M. D'Orbigny's death his whole museum was bought by the Jardin des Plantes, and in the year 1867, through the courtesy of M. Fischer, I had an opportunity of examining the original specimen there; and although it was in a very dilapidated condition, I had no difficulty in satisfying myself that it was a true Crinoid of a very peculiar type.

Professor Louis Agassiz called at Barbadoes in the "Hassler" in 1873, and he there saw a second specimen of *Holopus* in Sir Rawson Rawson's collection. Professor Agassiz intended to have published a full description of the specimen, which was lent to him for that purpose by Sir Rawson Rawson, but he was prevented from doing so by failing health, and after his death the figures which

he had prepared were published by Alexander Agassiz, with a short note by Count Pourtales, in the "Zoological Results of the 'Hassler' Expedition."

During the last few years three specimens of *Holopus rangii* have fallen into Sir Rawson Rawson's hands, and from these we will be able to give a pretty good account of the hind parts. All were brought up on fishermen's lines from deep water off Barbadoes. One is very complete in all important points, wanting only the two "bivial" arms, but retaining the mouth-valves. The second is a little larger; it wants the mouth-valves, and again the bivial arms; and with Sir Rawson Rawson's sanction I boiled this specimen down, to figure and describe the separate parts. The third specimen is quite perfect, the arms closely curled in, in their normal position when contracted; but it is very young, only about 8 m.m. in height. Besides the four examples mentioned I am aware of only another, which I have not yet seen; it was shown at the Philadelphia Exhibition, and was afterwards bought by the Museum of Comparative Zoology at Cambridge, Mass.

Holopus is distinguished from all other recent Crinoids by having the basal plates, and the first and probably also the second radials fused together, forming the wall of a tube-like body-chamber, which is cemented beneath to the foreign body to which the Crinoid is attached, by an irregularly expanded calcareous base. This mode of attachment also occurs in the fossil genus *Apiocrinus*, and in many other forms of the Apiocrinidæ and Cyathocrinidæ, but in these, of course, the cement matter is thrown out at the base of a jointed stem.

The upper portion of this hollow column expands slightly, and its thickened upper border is divided into five strongly-marked facets for the articulation of five arms. Each facet is traversed by a transverse articulating ridge, a little in front of which there is the mouth of the tube which lodges the sarcode axis of the joints, and a little behind its centre there is a somewhat longer aperture which appears to lead into the cancellated structure of the outer part of the wall. There are two large shallow muscular impressions on the surface of the facet on the proximal aspect of the transverse ridge. These facets, I conclude, represent the upper surfaces of second radials, but if so, they differ from the second

radials of all other recent Crinoids in being connected with the radial axillaries by a true muscular joint instead of by a syzygy. The alternative is that they may be the upper articulating surfaces of the first radials, in which case the next joints may be formed of the second and third radials coalesced, and the syzygy between them obliterated; or, finally, there may be only two radials. There is no trace of any further division of the wall of the column, and the cavity is continued contracting gradually to the bottom. A vertical mark, sometimes a groove and sometimes a ridge, runs from the centre of each articulating facet down the inside of the wall for about two-thirds of the depth of the cavity, when it is lost. The upper border of the cup, bearing the facets, is very irregular in thickness; and in all the specimens which I have seen, including D'Orbigny's, one side of the border is much thicker and considerably higher than the other side, and the three arms articulated to it are much larger than those articulated to the opposite side. There is thus a very marked division into "bivium" and "trivium," and consequently a bilateral symmetry underlies the radiated arrangement of the antimeres. Singularly enough, the specimen described by D'Orbigny was abnormal, only four arms being developed, a circumstance which no doubt greatly conduced to the doubt with which its determination as an echinoderm was received.

Five "radial-axillary" plates, three larger and two smaller, articulate by corresponding ridges and muscular impressions with the facets of the border of the cup, and each of these bears distally two facets, sloping outwards and downwards, for the insertion of the first brachials. The outer surfaces of the radial axillaries are very gibbous, thrown out into almost hemispherical projections, studded with low tubercles; a deep groove runs up the centre of the inner aspect of the joint, and the two sides send inwards very strong projecting processes, which abut against the corresponding processes of the contiguous joints on either side, and lock into them by a system of corresponding ridges and grooves, so that there appears to be little or no motion between these joints.

The radial-axillaries are each succeeded by two series of about eight similar, thick, wedge-shaped brachial joints, very convex externally, and giving off laterally on each side of the arm alternating, very flat broad pinnules each consisting of about six plate-

like joints. The brachials are also provided with strong lateral processes forming a wall on either side of the radial groove, and the sides of adjacent series of these first eight arm-joints are marked with corresponding grooves and ridges, so that, although from the presence of articulating ridges of varying degrees of obliquity, and of muscular impressions; the proximal portions of the arms must be capable of some motion, that motion would appear to be slight. After about the eighth the joints suddenly contract in size and become greatly compressed, and this narrow series extends to about sixteen in number, gradually tapering to the end of the arm.

At the bases of the arms, just above the edge of the cup, five thick calcareous bosses, each composed of the contiguous lateral processes of two radial-axillary joints, project interradially into the cup; and opposite these five rather large triangular plates, meeting in the centre of the disk, form a low pyramid covering the mouth. The oral plates are interradial, and the spaces between them radial corresponding with the arm-grooves.

D'Orbigny described the animal as possessing no anal opening, and this is probably the case, but the material is still too scanty to admit of the full examination of a complete specimen of the skeleton, and the soft parts are unknown.

All the specimens of *Holopus* which have been hitherto procured are in a very peculiar condition; the thick-walled foot, and massive, somewhat rudely shaped cup and arm-joints are formed of a loose spongy calcified areolar tissue deeply stained with a black-green pigment. There is no appearance of any separate organic matter, either on the outer surface of the skeleton, which is very delicately sculptured like shagreen, or on the articulating surfaces of separated plates; indeed, the whole body is so perfectly hard and rigid that at first sight I thought it might be semi-fossil. It is without doubt recent, but I suspect that the tissues are very imperfectly differentiated, almost protoplasmic. When an arm is put into boiling water it falls into pieces at once, the joints simply coming asunder, and showing no trace of muscular or other organic connection except the axial cords of the joints, which sometimes keep two joints hanging in connection for a little.

Holopus is thus specially characterised among living Crinoids

by the absence of an articulated stem or its representative the centro-dorsal plate; by its viscera being lodged in a hollow peduncle with a continuously calcified wall; and by the absence of an anal opening.

In 1846 Professor Steenstrup described under the name of *Cyathidium*, a genus of fossil Crinoid from the chalk of Faxoe, which occupies a debatable position between the base of the Eocene Tertiaries and the top of the Cretaceous series. The only portion yet described of *Cyathidium* is a deep cup or tube with a spreading base of attachment and a thickened rim with articulating facets for five arms. The cup of *Cyathidium* is somewhat more symmetrical and coralloid than that of the recent West Indian form; but I see no distinction between them of generic value, and I think we must accept *Holopus* as another of the links which recent investigations have made so numerous between the faunæ of later geological periods and that of the present time.

2. On the Diurnal Oscillations of the Barometer.—Part II. By Alex. Buchan, M.A.

In this communication the author stated that he limited his remarks on the present occasion to some of the more prominent results he has arrived at in the course of this investigation. The paper itself, with the tables, will be submitted when the computations have been finished and thoroughly revised,—a work which must necessarily yet take some considerable time.

It is proposed that Part II. consist chiefly of tables showing the arithmetic mean values of the hourly variations of the different months of the year, with remarks on the more evident conclusions which may be drawn from them. The number of places for which data of more or less completeness have now been obtained exceed 130, situated in different parts of the globe. To these it is proposed to add the results of observations made at sea, chiefly those made by the "Challenger" and "Novara" expeditions.

As regards temperate regions, such as Great Britain, periods of no more than three years' observations give only the broadest characteristics of the diurnal barometric curve. From 20 to 25 years will probably be found to be required to show the

peculiarities of the curve with such completeness as to exhibit the variations impressed on the curve by season and by geographical position, particularly as regards masses of land and extended sheets of water. In lower latitudes a comparatively short time is required; but even here, where a general regularity in the phenomena is perhaps the most striking fact in the meteorology of equatorial regions, the variations which do occur from year to year ought to be carefully observed from their important bearing on the whole theory of the movements of the atmosphere. The summer months of the northern hemisphere, as regards the diurnal oscillations of the barometer, that is the period when the influence of the sun is at the maximum as regards its effects on these phenomena, are May, June, and July, and the winter months, November, December, and January, both corresponding with the sun's declination; that is, the effects are not cumulative, as in the case of the temperature of the air or that of the sea, by which the critical periods are retarded from one to two months.

Among the many interesting features of the curves which were pointed out may be noted the enormous influence of latitude and of land and sea respectively in determining the amount and time of occurrence of the different phases of the oscillations, and a diagram was exhibited showing the curves of a large number of places, from which it appeared that as regards the summer the A.M. maximum occurs at any time from 6-7 A.M. to 2 P.M., and the P.M. minimum from 3 to 8 P.M.—the stations selected showing a regular gradation between these extremes, a gradation dependent on geographical position. The tendency of assimilation of the curves for certain elevated stations and those for strictly sea-side stations was pointed out, and attention was drawn to the striking fact that the summer curves of inland stations within lat. 30° N. and S. essentially differed from those of higher latitudes—a difference which the varying declination of the sun with season failed to obliterate.

An examination of the different theories yet propounded shows that none of them are in accordance with the facts which have been collected. It would not be difficult by a proper selection of stations to bring proof in support of any of these theories. The truth is, however, that as more facts are obtained the difficulty of framing a satisfactory theory can scarcely be said to be materially

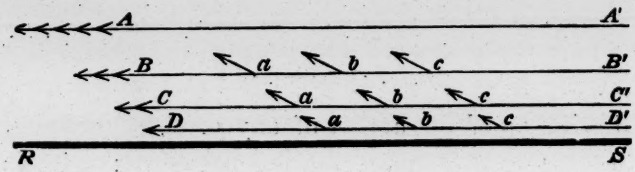
lessened, even though it becomes easy to arrive at a close approximation to the curves of a place from its geographical position alone, before determining the curves by working them out numerically.

An examination of these curves by the harmonic analysis and a similar examination of the temperature, hygrometric, wind, magnetic, and electric curves, will in all likelihood be required before the true theory of the diurnal barometric oscillations can be stated.

3. On the Air dissolved in Sea-Water. By J. Y. Buchanan.

4. Why the Barometer does not always indicate real Vertical Pressure: A continuation of the Paper laid before the Royal Society of Edinburgh in July 1875, in which, in addition to several other points, this was attempted to be shown. It is now more fully written out. By Robert Tennent.

The barometer only indicates real pressure when the atmosphere is in a state of perfect rest. It may then be represented a existing in vertical columns, but when it moves over a resisting surface its lower surface currents will be greatly retarded, while those aloft will move comparatively free and unimpeded. In this state it may be represented as moving in columns inclined in the direction towards which it moves. The atmosphere may thus be conceived as being divided into a number of spheroidal concentric layers, each of which is possessed of a different rate of speed, and moves more rapidly than the one beneath it: an increasing amount of friction will take place betwixt the layers as they approach the surface where its influence is greatest. What takes place may be represented in this way. Let R S be the resisting surface, and let



A A′, B B′, C C′, and D D′ represent different layers of air moving

at different rates of speed. Those being most rapid where the arrow heads are most numerous. Let these layers also exhibit equal masses of air, but of different volumes, which increase in size from the ground upwards, accompanied also by much greater mobility, as shown by Tyndall in his experiments at Chamouni and the summit of Mount Blanc. An important point is this. The surface current D D′ has only a horizontal source of supply from the direction D′ from which it is fed, while the upper current A A′ has not only a horizontal source of supply from A′, but it also derives supply from the slower moving current B B′ beneath it, which will be drawn upwards, or "lifted," in the direction of the small inclined arrows *a b c d*. To enable this upper current to supply itself in this way, while it possesses the same amount of mass as that beneath it, it must have a greater velocity, and therefore greater momentum than that of the current beneath it, from which it draws its supply. Each of the lower currents, as they approach the surface, are also supplied in the same manner, but in a decreasing ratio, from those beneath them, until the lowest layer is reached, which is that which is most retarded, not only owing to its proximity to a resisting surface, but also to the scarcity of supply, which can now only be derived from a horizontal source, and not from beneath, as was the case with those above it. A tendency in the air to accumulate aloft will now take place, by "lifting," in the direction of the small inclined arrows *a b c d*. Pressure is thus diminished at the surface, while it is abnormally increased higher up.

The effect of this will be that the surface barometer will exhibit diminished or fictitious pressure, while the real weight of the atmosphere remains unaltered. A partial vacuum is found on the lee-side of a wall, over the top of which a strong wind blows: it withdraws the air there to such an extent, that it causes removal to exceed restoration, but it does not affect the real vertical mass of the column overhead except to an infinitesimal degree. In this, as in the former case, real pressure cannot be ascertained by the surface barometer. It is only to be obtained from the result of an observation of a series of barometers placed vertically above each other, and not very far apart. From the results thus obtained, it would be found that the normal upward diminution of pressure

which takes place when the atmosphere is at rest would be greatly altered when its upper portion is in rapid motion.

Retarded surface currents and rapid upper currents which move in inclined columns and produce these effects, can only be found with an imperfect fluid, and on a resisting surface, into which the element of friction enters. On a frictionless surface this could not take place. The atmosphere would there move in vertical, not in inclined columns, no "lifting" would take place, pressure would be real or statical; its upward diminution would be normal, as when it is at rest, and horizontal movement would not take off vertical pressure.

Barometric pressure must hence be regarded in two points of view—1st as being a cause; and 2d, as being an effect. When, as in the first case, it is real or statical, it operates as a cause due to gravitation, which is unresisted. When, as in the second instance, there is an introduction of the dynamical element, surface gravitation is diminished by "lifting," and it must then to a certain extent be regarded as an effect. The practical conclusion from this is obvious. On weather charts the constant rise and fall of the barometer, which is there reported, *is to a large extent simply due to the passage of air over a resisting surface.* Over a surface devoid of friction these mechanical effects would be entirely removed: its rise and fall would be greatly reduced, and might be considered as being solely dependent on the effects of heat and vapour. The gradients and isobars which are represented by these movements of the barometer would consequently require also to be similarly corrected.

The barometer does not indicate the real weight of the atmosphere, it only exhibits the amount of its elasticity, from which its real weight can only be deduced when the dynamical element of motion does not enter into its currents. The two cases above mentioned may illustrate this point. The surface barometer there indicates fictitious pressure, or in other words, *the amount of pressure due to the elasticity of the air, but not to that of its real weight,* which is there diminished by "lifting," and as lifting can increase or diminish in amount, so also can the elasticity of the air, while its real weight remains unaltered.

As a general rule, in the British Isles, equatorial winds are

accompanied by these rapid upper movements, while polar winds move with a greater uniformity in the velocity of their various layers, and sometimes even those on the surface move more rapidly when copiously supplied from a vertical source. There is thus a *remarkable difference in their mode of inflow.* Equatorial winds, as they increase in force, are hence accompanied by "lifting" and a fall of the barometer. Polar winds are not attended by "lifting," and if their supply is copious and partly from a vertical source, their increase in force is accompanied by a rise of the barometer.

The range of the thermometer is equally great, both above and below its mean; but with the barometer the extent of its range above the mean is not more than one-half of that which takes place when it is below it. When it is below the mean, equatorial winds generally prevail, which are accompanied by lifting and extensive range. When above the mean, polar winds prevail, which are not attended by lifting or such extensive fluctuations. Hence, as a general rule, equatorial winds exhibit fictitious or dynamical pressure, while polar winds possess more nearly, real or statical pressure, being less accompanied by rapid upper currents, and by the mechanical oscillations due to the passage of air over a resisting surface.

Observations of a general description and illustrations will probably be afterwards introduced to exemplify the above conclusions.

It is to this *difference in the mode of inflow* that an attempt has been made to explain the causes why depressions move in an easterly direction.

5. Laboratory Notes. By Professor Tait.

(*a*) On an Effect of Heat on Electro-Static Action.

By means of a very delicate galvanometer, transient currents were detected when one of two plates of the same, or of different, metals, separated by a sheet of mica or glass, was suddenly heated.

(*b*) On Dr Blair's *Scientific Aphorisms* in connection with the Ultra-Mundane Particles of Le Sage.

Accident has recently called my attention to a work entitled *Essays on Scientific Subjects*, by Robert Blair, Regius Professor of

Practical Astronomy in the University of Edin. (Edin. 1818). In the University Library there is a second edition of a part of the same work with the title *Scientific Aphorisms* (Edin. 1827). I bring them before the notice of the Society, as they contain an explanation of gravitation, &c., almost identical with that of Le Sage, to which our attention was lately recalled by our President. Professor Blair seems to have invented this explanation for himself—because, though he gives frequent references to other authors, whose results he quotes, he makes, so far as I have seen, no reference to Le Sage.

On a future occasion I may enter on a discussion of the points of resemblance and difference of these two theories.

5. Note on an Identity. By Professor Tait.

Whatever be p and q it is obvious that

$$\frac{1}{p} = \frac{1}{q} + \frac{q-p}{q} \cdot \frac{1}{p}$$

Hence
$$\frac{1}{p} = \frac{1}{q_1} + \frac{q_1-p}{q_1}\left(\frac{1}{q_2} + \frac{q_2-p}{q_2} \cdot \frac{1}{p}\right)$$

and so on. Finally we see that

$$\frac{1}{p} = \frac{1}{q_1} + \frac{q_1-p}{q_1} \cdot \frac{1}{q_2} + \frac{q_1-p}{q_1} \cdot \frac{q_2-p}{q_2} \cdot \frac{1}{q_3} + \ldots\ldots$$

$$\ldots + \frac{q_1-p}{q_1} \cdot \frac{q_2-p}{q_2} \ldots \frac{q_{n-1}-p}{q_{n-1}} \cdot \frac{1}{q_n} + \frac{q_1-p}{q_1} \cdot \frac{q_2-p}{q_2} \ldots \frac{q_n-p}{q_n} \cdot \frac{1}{p} .$$

absolutely without any restriction on the values of the quantities involved.

It is obvious that an immense number of curious results in the form of sums of series, &c. can be derived with great ease from this expression and from various modifications of it. I give, therefore, only a few very simple examples.

Take q_1, q_2, &c., as the first n of the natural numbers, and the series becomes

$$\frac{1}{p} = 1 - \frac{p-1}{2} + \frac{p-1}{2} \cdot \frac{p-2}{3} - \ldots .$$

$$(-)^{n-1} \frac{p-1}{2} \cdot \frac{p-2}{3} \ldots \frac{p-n-1}{n} (-)^n \frac{p-1}{1} \cdot \frac{p-2}{2} \ldots \frac{p-n}{n} \frac{1}{p},$$

whence at once the sum of the first $n+1$ terms of the expansion of $(1-1)^p$ is seen to be

$$(-)^n \frac{p-1}{1} \cdot \frac{p-2}{2} \cdots \cdot \frac{p-n}{n}.$$

We obtain merely the same result if we take q_1, q_2, &c., as any set of consecutive whole numbers; but from the theorem itself it is easy to obtain the equality,

$$\frac{p}{r}\left\{1 + \frac{p+r}{r+1} + \frac{p+r}{r+1} \cdot \frac{p+r+1}{r+2} + \cdots + \frac{p+r}{r+1} \cdots \frac{p+s-1}{s}\right\}$$

$$= \frac{p+r}{r} \cdot \frac{p+r+1}{r+1} \cdots \frac{p+s}{s}.$$

Next, write the general identity as follows:—

$$\frac{1}{p} = \frac{1}{q_1} + \frac{p}{q_2}\left(\frac{1}{p} - \frac{1}{q_1}\right) + \frac{p^2}{q_3}\left(\frac{1}{p} - \frac{1}{q_1}\right)\left(\frac{1}{p} - \frac{1}{q_2}\right) + \cdots$$

$$+ \frac{p^{n-1}}{q_n}\left(\frac{1}{p} - \frac{1}{q_1}\right) \cdots \left(\frac{1}{p} - \frac{1}{q_{n-1}}\right) + p^{n-1}\left(\frac{1}{p} - \frac{1}{q_1}\right) \cdots \left(\frac{1}{p} - \frac{1}{q_n}\right).$$

If in this we write each letter for its reciprocal we have

$$p = q_1 + \frac{q_2}{p}(p - q_1) + \frac{q_3}{p^2}(p - q_1)(p - q_2) + \text{\&c.},$$

of which a particular case is the curious formula

$$p = 1 + 2\left(1 - \frac{1}{p}\right) + 3\left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right) + \cdots$$

$$+ n\left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right) \cdots \left(1 - \frac{n-1}{p}\right) + p\left(1 - \frac{1}{p}\right) \cdots \left(1 - \frac{n}{p}\right).$$

Another is

$$1 = \cos\theta + \cos 2\theta(1 - \cos\theta) + \cos 3\theta(1 - \cos\theta)(1 - \cos 2\theta) + \ldots$$

$$+ \cos n\theta(1 - \cos\theta) \ldots (1 - \cos(n-1)\theta)$$

$$+ (1 - \cos\theta)(1 - \cos 2\theta) \ldots (1 - \cos n\theta),$$

of which a very interesting case is given by $n\theta = 2\pi$.

As a final example we have the singular for mula,

$$\frac{1}{x-y} = \frac{1}{x} + \frac{y}{x(x+1)} + \frac{y(y+1)}{x(x+1)(x+2)} + \ldots\ldots \text{ \&c.}$$

whence it follows that, subject to the introduction of the remainders as above (which vanish if the series are extended to infinity, and if $x > y$),

$$\left(\frac{1}{x}+\frac{y}{x(x+1)}+\frac{y(y+1)}{x(x+1)(x+2)}+\ldots\right)\left(\frac{1}{x}-\frac{y}{x(x+1)}+\frac{y(y-1)}{x(x+1)(x+2)}+\ldots\right)$$

$$=\frac{1}{x^2}+\frac{y^2}{x^2(x^2+1)}+\frac{y^2(y^2+1)}{x^2(x^2+1)(x^2+2)}+\ldots\ldots$$

By another application of the formula we may easily obtain finite expressions for the sum of the series of which two successive terms are

$$\frac{y(y+1)\ldots.(y+r-1)}{x(x+1)\ldots.(x+s-1)}. \text{ and } \frac{y(y+1)\ldots.(y+r)}{x(x+1)\ldots.(x+s)}.$$

I obtained the first expression above by integrating *by parts* a power such as x^{p-1}, but the following mode of obtaining it shows at once its nature.

Let there be a number of independent events, A, B, ... N, whose separate probabilities are α, β, ... ν. Then the chance that one at least of them occurs is

$$1-(1-\alpha)(1-\beta)\ldots.(1-\nu).$$

But we may obtain another expression for the same result by writing the chance that any one (say A) occurs, adding to that the chance that another (say B) occurs while A does not occur, then that C occurs and neither A nor B, &c. This gives

$$\alpha+\beta(1-\alpha)+\gamma(1-\alpha)(1-\beta)+\ldots.$$

Equating these two expressions we get an identity which is easily transformed into that first given.

But its truth is much more easily seen if we write α' for $(1-\alpha)$, &c., when the last given form becomes

$$1-\alpha'\beta'\gamma'\ldots.=1-\alpha'+(1-\beta')\alpha'+(1-\gamma')\alpha'\beta'+\ldots.$$

which is an obvious truism. The method seems well worth the attention of any one with leisure and some analytical skill.

July 24.—Mr Muir has kindly given me a reference to Crelle, vol. xii. p. 354, where it is stated that the above identity in one of its forms is in Schweins' "Analyse," p. 237. This work I have not seen. Mr Muir adds that no developments or applications of the theorem are made.

The following Gentlemen were duly elected Fellows of the Society:—

1. James Stevenson, 4 Woodside Crescent, Glasgow.
2. James Roberton, LL.D., Professor of Conveyancing, 1 Park Terrace East, Glasgow.
3. George A. Panton, 24 Bennet's Hill, Birmingham.
4. Isaac Bayley Balfour, Sc.D., 27 Inverleith Row.
5. Sir Daniel Macnee, 6 Learmonth Terrace.
6. William Pole, Mus.Doc., Memb. Inst. Civil Engineers, 31 Parliament Street, Westminster.